Embedding the Petersen Graph on the Cross Cap

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April 8, 2014

Introduction

In this project we will introduce the *Petersen graph* and highlight some of its interesting properties, explain the construction of the *cross cap* and proceed to show how to embed the Petersen graph onto the suface of the cross cap without any edge intersection – an embedding that is not possible to achieve in the plane.

Since the self-intersection of the cross cap in $\mathbb{R}^3$ is very hard to grasp in a planar setting, we will subsequently create an animation using the 3D modeling software *Maya* that visualizes the important aspects of this construction. In particular, this visualization makes possible to develop an intuition of the object at hand.

1 Theoretical Preliminaries

We will first explore the construction of the Petersen graph and the cross cap, highlighting interesting properties. We will then proceed to motivate the embedding of the Petersen graph on the surface of the cross cap.

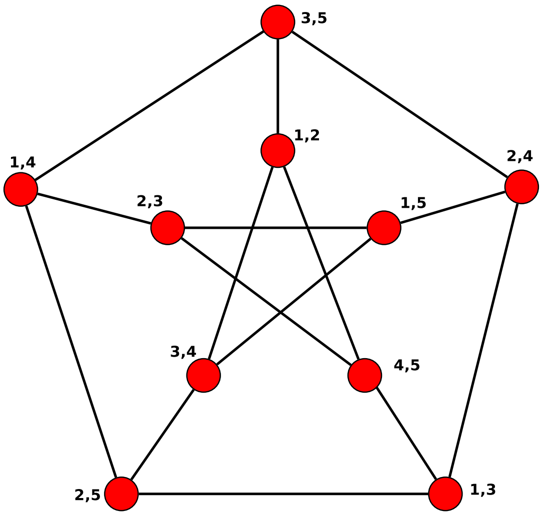
1.1 The Petersen Graph

The Petersen graph is an important example from graph theory that has proven to be useful in particular as a counterexample. It is most easily described as the special case of the *Kneser graph*:

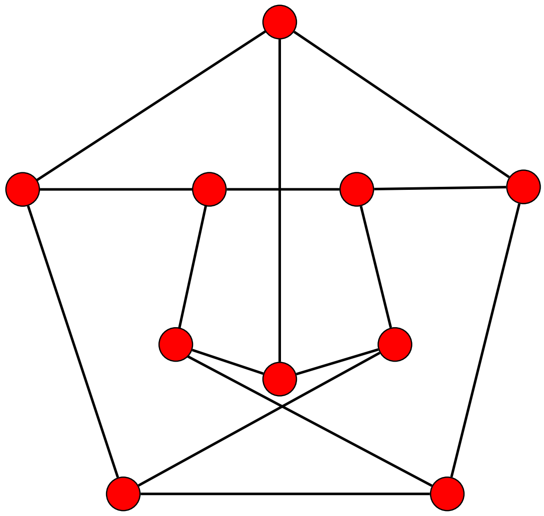
Definition 1 (Kneser graph)**.** The *Kneser graph of n, k* (denoted $KG_{n,k}$) consists of the k-element subsets of $\{1, \dots, n\}$ as vertices, where an edge connects two vertices if and only if the two sets corresponding to the vertices are disjoint.

The graph $KG_{n,k}$ is $\binom{n-k}{k}$-regular, i. e. at each vertex, $\binom{n-k}{k}$ edges meet. The special case we will focus on is the Petersen graph $P = KG_{5,2}$, so called after the Danish mathematician Julius Petersen (1839–1910). It is 3-regular and has 10 vertices and 15 edges. See figures 1a and 1b.

The Petersen graph has the following interesting properties [3, 5]:

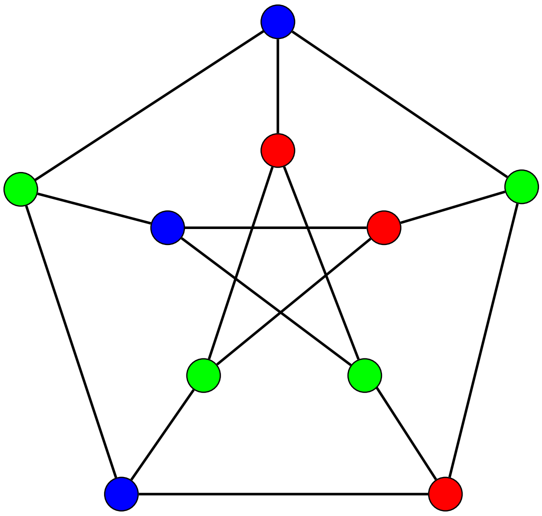


(a) Traditional realization of the Petersen graph as $KG_{5,2}$ in the plane. The inner vertices and their edges form a pentagram; the outer vertices form a pentagon. This version exhibits five edge intersections.

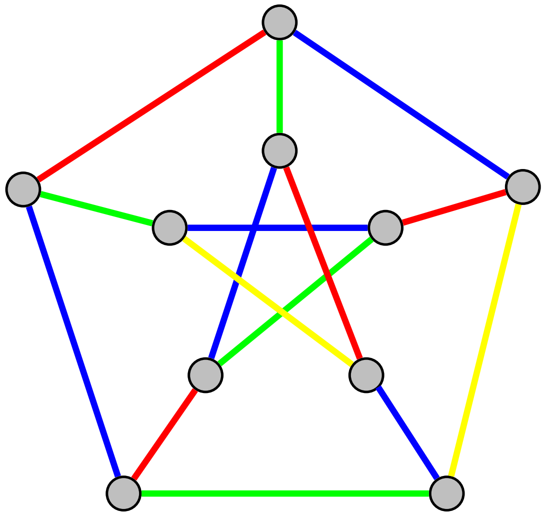


(b) The Petersen graph in the plane with just two edge intersections. It can be shown that this is the minimum number of intersections necessary when drawing the graph in the plane.

Figure 1: Two drawings of the Petersen graph in the plane



(a) A vertex 3-coloring of the Petersen graph



(b) An edge 4-coloring of the Petersen graph

Figure 2: The Petersen graph has chromatic number 3 and chromatic index 4

- It has chromatic number $n - 2k + 2 = 3$, so an optimal vertex coloring needs at least three different colors (see figure 2a)
- It has chromatic index 4, meaning a minimum edge coloring must use at least four different colors (see 2b)
- It is the smallest *snark*[1] – a connected, bridgeless[2], 3-regular graph with chromatic index 4
- It has crossing number $\mathrm{cr}(P) = 2$, meaning it cannot be embedded into the Euclidean plane $\mathbb{R}^2$ with less than 2 edge intersections (cf. [7, p. 2], see figure 1b).

The Petersen graph can, however, be embedded without any edge intersections in the real projective plane, a model of which is the cross cap. While there are intuitive visualizations in the Euclidean plane for the graph-theoretical properties, the embedding on the cross cap should be done in a 3D setting. This will be done in section 2.

1.2 The Real Projective Plane and the Cross Cap

Definition 2 (Real projective space)**.** The *real projective space* $\mathbb{R}P^n$ consists of the lines passing through the origin of $\mathbb{R}^{n+1}$. In the case $n = 1$, this is called the *real projective line*; in the case $n = 2$, *real projective plane*.

An equivalent (and more intuitive) construction for $\mathbb{R}P^n$ can be given by identifying antipodal points of the n-sphere S^n: Since any line passing through the origin meets the sphere at exactly two antipodal points, one can identify a line by either of these points. Choosing w. l. o. g. the points on the northern hemisphere, one finds that the real projective plane is topologically equivalent to the disk D^2 with antipodal points of the border $\partial D^2 = S^1$ identified.

Stretching the imagination a little bit, we can picture D^2 as the topologically equivalent unit square. The identification of antipodal border points then happens as in figure 3.

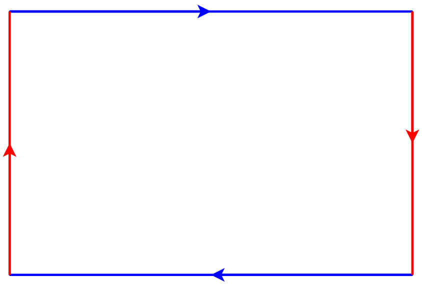


Figure 3: The topological view: How to glue $[0,1] \times [0,1]$ together to construct the real projective plane.

The homology of $\mathbb{R}P^n$ is [2, 2.42]:

$$H_k(\mathbb{R}P^n;\ \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k = 0 \text{ or } k = n \text{ odd} \\ \mathbb{Z}/2\mathbb{Z} & \text{if } k \text{ odd and } 0 < k < n \\ 0 & \text{otherwise} \end{cases}$$

As a corollary of the Alexander duality [2, 3.45], any compact and locally contractible subspace of $\mathbb{R}^n$ is torsion-free in homology of degree $n - 2$. But $H_1(\mathbb{R}P^2;\ \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$, hence the real projective plane cannot be embedded into $\mathbb{R}^3$; it can only be *immersed*, i. e. it can be locally embedded at any point.

[1] Julius Petersen constructed this graph in order to show that not all connected, bridgeless cubic graphs have chromatic index 3. In fact the Petersen graph, constructed in 1898 by Petersen, remained the only known counterexample until 1946 (cf. [3, p. 71]).

[2] Removing an arbitrary edge will not break the connectedness of the graph.

The *cross cap* is a two-dimensional real manifold that is homeomorphic to the real projective plane $\mathbb{R}P^2$. It will serve as our model of the real projective plane in $\mathbb{R}^3$.

1.3 Embedding the Petersen Graph on the Cross Cap

Using this "planar model" of $\mathbb{R}P^2$, we can now easily see that an embedding of the Petersen graph without edge intersection onto the surface of the cross cap is in fact possible. To see this, start with the Petersen graph in the form with just two edge intersections (figure 1b), leaving out all the edges that would intersect. Embedded in $\mathbb{R}P^2$ (figure 3) one now draws the remaining edges in the following fashion, taking the dotted line as an example (cf. figure 4): Draw the edge to leave the $[0,1] \times [0,1]$ square (with origin at the lower left, say) at $(0, 1/4)$. Because the red sides are glued together with opposite orientation, continuing to draw this edge will commence from $(1, 3/4)$. In the same fashion, the dashed edge passes the points $(0, 3/4)$ and $(1, 1/4)$ simultaneously (because they are in fact the same point in this construction); the same goes for the solid line at the points $(0, 1/2)$ and $(1, 1/2)$. Neither of these points induce a crossing.

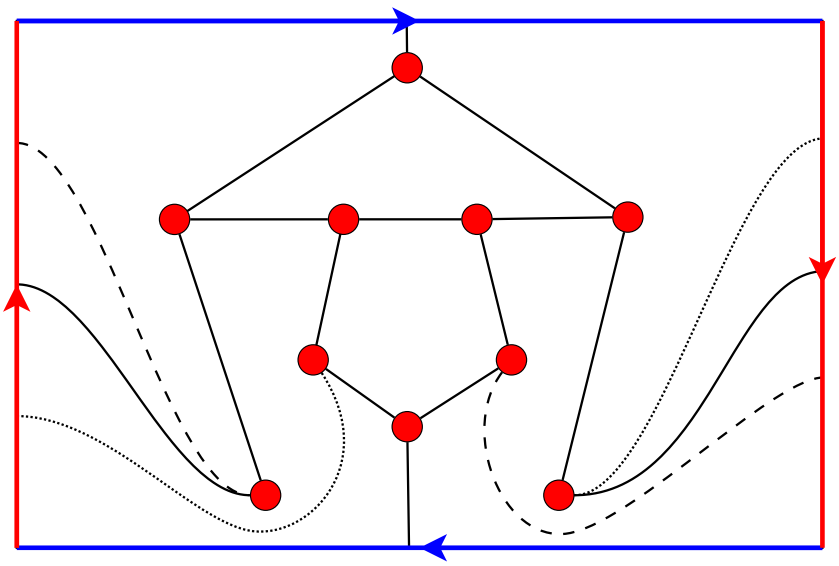


Figure 4: An embedding of the Petersen graph without edge intersection on the surface of the cross cap.

This result shows that it is in fact possible to embed the Petersen graph onto the cross cap without any edge intersection – however, this model of the cross cap has hardly any resemblance of how a model of it would look like immersed in $\mathbb{R}^3$.

1.4 An Alternative Embedding Approach Using the Dodecahedron

It is possible to obtain the embedding described in the previous section along with a construction of the cross cap "in one step". The *(regular) dodecahedron* (from gr. δωδεκα, twelve) is the platonic solid that consists of 12 regular pentagonal faces. Three of these faces meet at each vertex, hence the graph consisting of the 20 vertices and 30 edges is cubic.

One can now transfer the construction of $\mathbb{R}P^2$ from S^2 to the dodecahedron: Identifying antipodal points, we get a surface that consists of 6 faces and whose 10 vertices and 15 edges form the cubic Petersen graph. While this construction is very neat and elegant, it does not make clear how the Petersen graph appears on the surface.

2 The *Maya* Construction

Given that it is hard to grasp the "actual look" of the cross cap in its planar form, we realized it as a real 2-manifold embedded in $\mathbb{R}^3$. To do this, we chose the 3D modeling software *Autodesk Maya*® (or simply *Maya*). It is a state-of-the-art system not only for modeling objects, but also for creating animations with these objects.

Our goal was to create an insightful animation that highlights the following points we previously outlined in section 1:

- How the cross cap can be immersed in Euclidean space $\mathbb{R}^3$
- How the immersion will have self-intersections
- How the Petersen graph can be embedded onto this surface
- How this embedding will not have edge intersections

In the following sections we will present our motivation for designing the animation the way we did, and give a brief coverage of the technical steps that were necessary in *Maya* to produce this result.

2.1 Constructing the Cross Cap

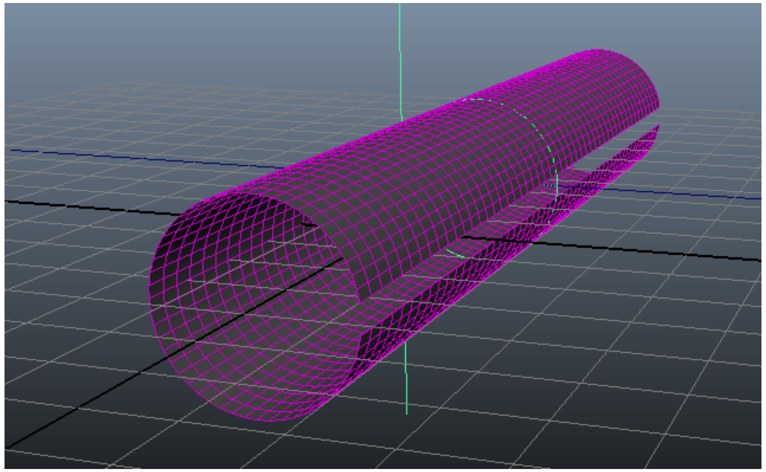
Initially we tried a construction of the cross cap starting with a dodecahedron as outlined in section 1.4 and motivated by [1, p. 270f]. A standard topological construction of point identification is possible to achieve vertex or edge-wise: Analogously to the construction of $\mathbb{R}P^2$ from S^2, one can start off with one half of the dodecahedron and continue to identify border edges and vertices, bending and inflating the object in the process. The advantage of this process is that it is producing the Petersen graph "for free" from the original edges.

This construction process works well in practice – however the resulting object is very difficult to comprehend in 3-space. So despite losing the advantage of the dodecahedron construction method, we chose the cross cap as depicted in [6, section 1.7] as this model is more symmetric and gives a better idea of the object itself.

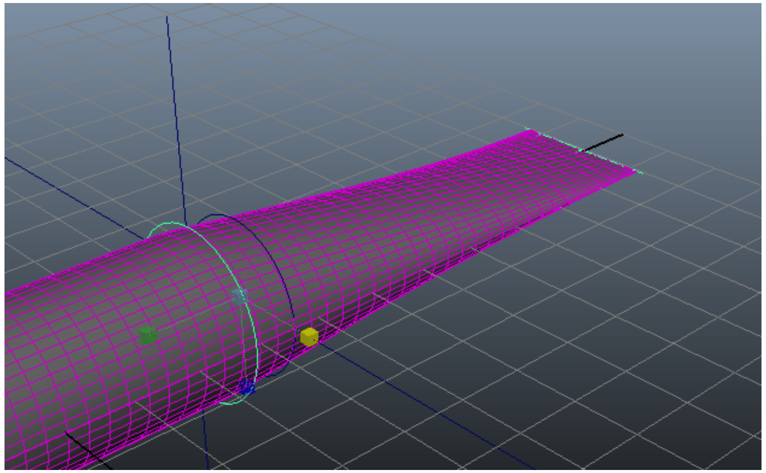
To model a symmetric cross cap, instead of employing a parametrization, we used the readily available modeling tools of Maya. Starting of with a highly subdivided grid, we bent and flared it in the right directions, getting an object very close to the parametrized cross cap (cf. [6]). For the construction process, see figures 5a–d on page 6.

In order to make the self-intersection of the cross cap visible, we designed a texture for our surface in such a way that it will make our surface appear to have an actual self-intersection.

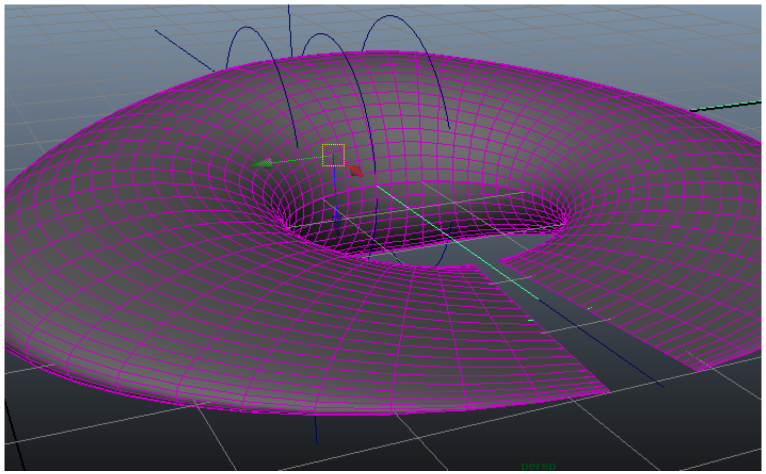
Figure 5: The construction process of the cross cap model



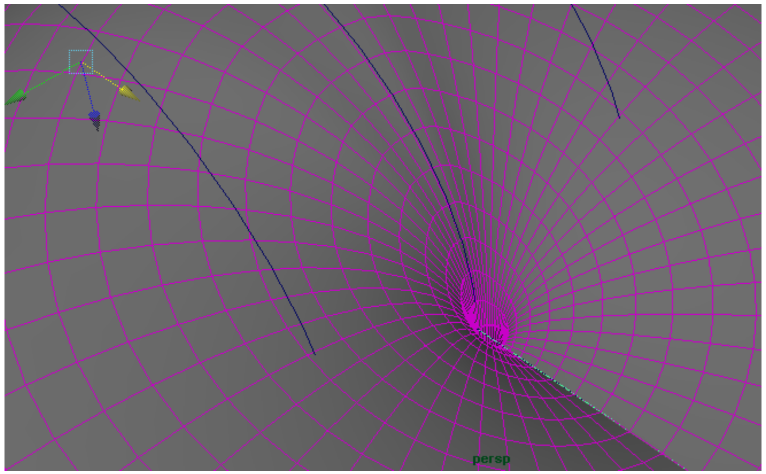
(a) Starting with a highly subdivided polygonal surface, bend it to a tube with the *Deformation / Nonlinear Bend* tool.



(b) Press down both sides of the tube using the *Deformation / Nonlinear Flare* tool. Adjust the parameters of the tool so that the middle still bulges.



(c) Applying another *Deformation / Nonlinear Bend* around the center, bring together the two “flattened” ends of the tube.

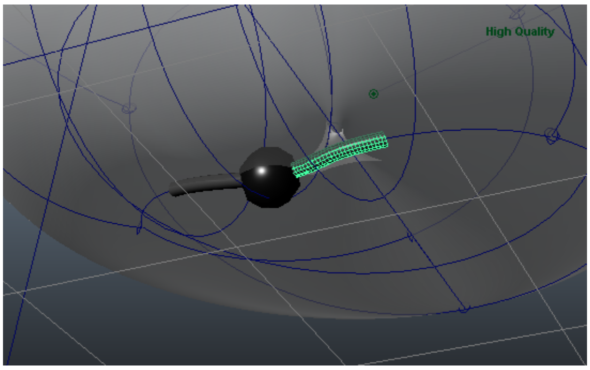


(d) Moving the bend handle from step (c), “close” the hole in the middle of the figure.

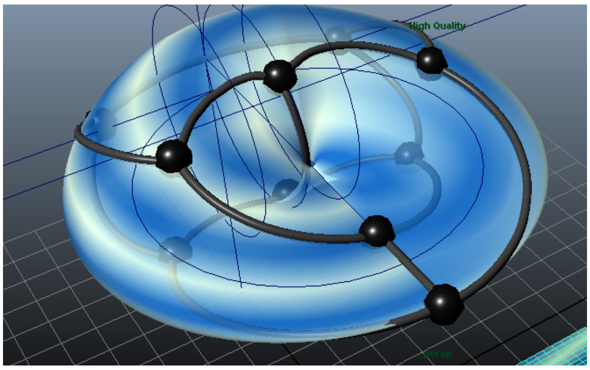
2.2 Embedding the Petersen Graph

To add the Petersen graph we projected suitable curves on the surface, according to the construction described in section 1.3 – see figure 7a. At the self-intersection, we manually chose on which part of the surface the curve will continue. We subsequently thickened them, and in particular used sphere primitives to indicate the vertices of the graph.

Figure 6: The final construction steps: Drawing the Petersen graph and applying a texture



(a) Using the *Make selected object live* switch, draw cubic EP curves on the surface. Create a NURBS circle primitive orthogonal to the line drawn and use the *Surface / Extrude* tool: This will produce the cartesian product of the circle along the curve. A slightly bigger sphere primitive will serve as vertex of the graph.



(b) The surface is still a plane bent to look like a closed 2-manifold. Apply a suitable texture to make clear how the self-intersection happens, even though the surface itself does not actually self-intersect. The texture should be half-transparent to give a good view of all vertices and edges, regardless of their position.

2.3 Designing an Animation

The first part of our animation shows the cross cap being viewed by a camera going around the object. By seeing an object in (passive) motion the viewer is able to get an advanced picture of the focussed object. The route of the camera is chosen such that the viewer will get a glance from all important perspectives and be able to make a whole three-dimensional picture on her own.

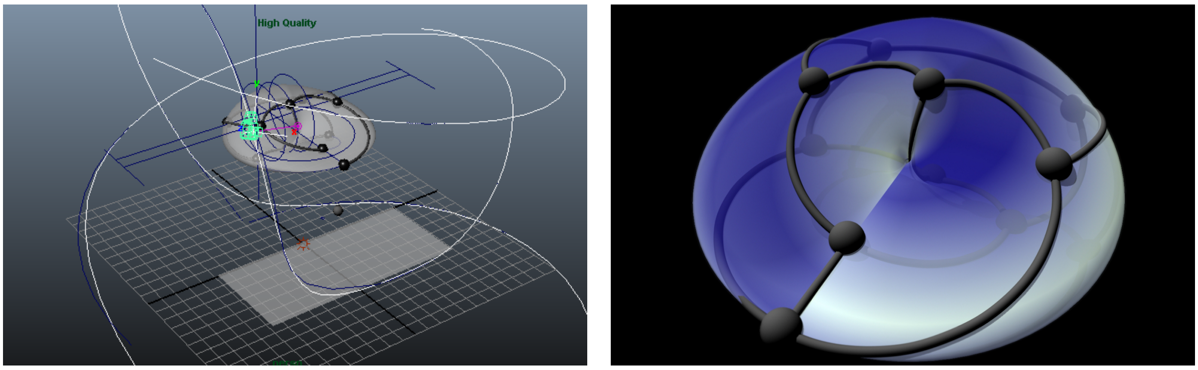
A special highlight is that we chose the texture to indicate the necessary self-intersection the object exhibits when immersed in three dimensional space. This way, the viewer immediately sees that at the intersection, the otherwise smooth gradient coloring of the surface object suddenly breaks. At second glance she will see how the coloring actually continues smoothly, but on the previously obstructed "underside".

In the second part of our animation, we show how the Petersen graph is drawn on the surface of the cross cap. As motivation for the drawing order we use figure 4, first drawing the lines in the middle that can be drawn without self intersection of edges even in the plane.

Then the camera makes moves, focussing on how each of the remaining edges is drawn sequentially on the surface without producing any self-intersection.

You can view an online version of the rendered video here: `http://feh.userpage.fu-berlin.de/petersen.html`

Figure 8: The camera motion and a rendered image



(a) A global view of the camera moving around the object. The rectangle at the bottom is used to view the texture of the cross cap in the plain.

(b) A screenshot from the resulting video. We use the Maya's raytracing mode in order to get realistic-looking shadows and transparency effects.

3 Conclusion

We introduced first the Petersen graph and then the cross cap; we then showed how the graph can be embedded on the surface of the cross cap. With this mathematical motivation we set out to create a visualization of this process in 3-space. By moving the point of view around the object several times, the viewer can form an intuition about the construction in her mind.

References

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