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Master's Thesis

Realization of Cyclic Spaces

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Nur wenn man nicht auf den Nutzen nach außen sieht, sondern in der Mathematik selbst auf das Verhältnis der unbenutzten Teile, bemerkt man das andere und eigentliche Gesicht dieser Wissenschaft. Es ist nicht zweckbedacht, sondern unökonomisch und leidenschaftlich. – Der gewöhnliche Mensch braucht von ihr nicht viel mehr als er in der Elementarschule lernt; der Ingenieur nur so viel, daß er sich in den Formelsammlungen eines technischen Taschenbuches zurechtfindet, was nicht viel ist; selbst der Physiker arbeitet gewöhnlich mit wenig differenzierten mathematischen Mitteln. Brauchen sie es einmal anders, so sind sie zumeist auf sich selbst angewiesen, weil den Mathematiker solche Adaptierungsarbeiten wenig interessieren. So kommt es, daß Spezialisten für manche praktisch wichtigen Teile der Mathematik Nichtmathematiker sind. Daneben aber liegen unermeßliche Gebiete, die nur für den Mathematiker da sind: ein ungeheures Nervengeflecht hat sich um die Ausgangspunkte einiger weniger Muskeln angesammelt. Irgendwo innen arbeitet der einzelne Mathematiker und seine Fenster gehen nicht nach außen, sondern auf die Nachbarräume.

— Robert Musil, *Der mathematische Mensch* (1913)

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Introduction

The following text is a detailed account of the fundamental concepts and basic results in the theory of simplicial and cyclic spaces. Much of the literature concerns itself mainly with simplicial and cyclic *sets*; in this text, particular care has been taken to make statements about simplicial and cyclic *spaces*, which offer a somewhat richer structure.

Section 1 defines the simplicial and cyclic categories, which serve to impose a certain structure on the collection of spaces we study. Sections 2–4 define and study the realization functor, which uses the simplicial structure information to glue together the simplices of simplicial spaces to obtain a single topological space. Section 5 covers the most important aspect of cyclic spaces, namely that their realization admits a canonical circle action. A natural question to ask is how one can compute fixed points under this circle action (and subgroups thereof); answering this calls for an elaborate subdivision procedure, which is described in section 6. Finally the results of the preceding sections are used in section 7 to study the free loop space of the classifying space of a (topological) group.

The account given here thematically follows the first and second section of [BHM93], proving in detail many assertions made there; especially in the first sections, the quite modern and elementary treatment [Lod98] is also used.

1 The Simplicial and Cyclic Category

We introduce the simplicial category Δ, extend it to obtain Connes's cyclic category Λ and provide insightful examples.

Definition 1.1. The *n-dimensional standard simplex* $|\Delta^n|$ is defined as the subspace

$$|\Delta^n| := \left\{ (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \,\middle|\, x_i \geq 0,\ \sum_i x_i = 1 \right\} \subset \mathbb{R}^{n+1}.$$

Setting any $x_i = 0$ yields obvious inclusions $D_i : |\Delta^{n-1}| \hookrightarrow |\Delta^n|$ of the *i-th face*, i. e. the face opposite of the vertex $v_i = (0, \dots, 0, 1, 0, \dots, 0)$, which has as only nonzero entry the i-th coordinate. The maps D_i, $i = 0, \dots, n$, are called *coface maps.*

The *codegeneracy maps* $S_i : |\Delta^{n+1}| \twoheadrightarrow |\Delta^n|$, $i = 0, \dots, n$, are the linear projections such that $S_i(v_{i+1}) = v_i$. ⌟

Remark 1.2. It is clear that the behaviour of the coface and codegeneracy maps can be captured by observing what they do on the spanning vertices v_i of the simplex. We have obvious identities $D_j D_i = D_i D_{j-1}$ if $i < j$ and $S_j S_i = S_i S_{j+1}$ if $i \leq j$. When $i = j$ or $i = j + 1$, we also have $S_j D_i = \mathrm{id}$. If $i \neq j$, we have $S_j D_i = D_i S_{j-1}$ if $i < j$, and $S_j D_i = D_{i-1} S_j$ if $i > j + 1$. ⌟

This behaviour can be captured abstractly in the following definition (cf. [Lod98, 6.1.1]):

Definition 1.3. The *simplicial category* Δ has as objects the finite ordered sets $[n] := \{0, 1, \dots, n\}$, $n \in \mathbb{N}$. The morphisms are generated by the *coface maps* $\delta_i : [n-1] \to [n]$, $i = 0, \dots, n$ and the *codegeneracy maps* $\sigma_i : [n+1] \to [n]$, $i = 0, \dots, n$, subject to the following *cosimplicial relations*:

$$\begin{aligned} \delta_j \delta_i &= \delta_i \delta_{j-1} \text{ for } i < j \\ \sigma_j \sigma_i &= \sigma_i \sigma_{j+1} \text{ for } i \leq j \\ \sigma_j \delta_i &= \begin{cases} \delta_i \sigma_{j-1} & \text{for } i < j \\ \mathrm{id} & \text{for } i = j, j+1 \\ \delta_{i-1} \sigma_j & \text{for } i > j+1 \end{cases} \end{aligned} \tag{1}$$

In the opposite category Δ^{op} we write $d_i = \delta_i^{\mathsf{op}} : [n] \to [n-1]$ for the *face maps* and $s_i = \sigma_i^{\mathsf{op}} : [n] \to [n+1]$ for the *degeneracy maps*. The relations (1) translate in Δ^{op} to the following *simplicial relations*:

$$
\begin{aligned}
d_i d_j &= d_{j-1} d_i \text{ for } i < j \\
s_i s_j &= s_{j+1} s_i \text{ for } i \leq j \\
d_i s_j &= \begin{cases} s_{j-1} d_i & \text{for } i < j \\ \mathrm{id} & \text{for } i = j, j+1 \\ s_j d_{i-1} & \text{for } i > j+1 \end{cases}
\end{aligned}
\tag{1^{op}}
$$

A *simplicial set* $X_\bullet$ is a functor $X_\bullet : \Delta^{\mathsf{op}} \to \mathsf{Set}$. For brevity, we write $X_n := X([n])$ and denote by d_i and s_i the images of the respective maps under X. The elements of X_n are sometimes called *n-cells* of $X_\bullet$ (in particular, 0-cells are called *vertices*). An n-cell x is called *degenerate* if it can be obtained from an $(n-1)$-cell y as $x = s_i y$ for some i; else it is called *nondegenerate*.

The *category of simplicial sets* is the functor category $\mathrm{Func}(\Delta^{\mathsf{op}}, \mathsf{Set})$ and will be denoted by S. It has as objects simplicial sets, i. e. functors $X_\bullet, Y_\bullet : \Delta^{\mathsf{op}} \to \mathsf{Set}$, and as morphisms natural transformations $f_\bullet : X_\bullet \to Y_\bullet$ between these functors. ⌟

Example 1.4. In particular, Hom-sets are sets. We have a rather important example, the *standard n-dimensional simplicial set* Δ^n with $\Delta^n(-) = \mathrm{Hom}_\Delta(-, [n]) = \mathrm{Hom}_{\Delta^{\mathsf{op}}}([n], -)$. For a simplicial set $X_\bullet$, by the Yoneda lemma (cf. [Awo10, Lemma 8.2]) we have an isomorphism

$$\mathrm{Hom}_{\mathsf{S}}(\Delta^n, X_\bullet) = \mathrm{Nat}(\Delta^n, X_\bullet) \cong X_n$$

that is natural in both n and $X_\bullet$. Hence simplicial maps out of Δ^n are uniquely determined by X_n. ⌟

Example 1.5. We have already seen an example of a *cosimplicial space* in definition 1.1: It is a functor $|\Delta^\bullet| : \Delta \to \mathsf{Top}$ that assigns $|\Delta^{[n]}| = |\Delta^n|$, and sends $|\Delta^{\delta_i}| = D_i$, $|\Delta^{\sigma_i}| = S_i$.

We define the functor $\mathrm{Sing}(-) : \mathsf{Top} \to \mathsf{S}$ as follows: For every topological space Y, we set $\mathrm{Sing}(Y)(-) = \mathrm{Hom}_{\mathsf{Top}}(|\Delta^\bullet|, Y)$; continuous maps $g : Y \to Z$ become postcomposition with g. Given $f \in X_n = \mathrm{Hom}_{\mathsf{Top}}(|\Delta^n|, Y)$, we have

$$
\begin{aligned}
d_i(f) &= |\Delta^{n-1}| \overset{D_i}{\hookrightarrow} |\Delta^n| \overset{f}{\to} Y \in \mathrm{Hom}_{\mathsf{Top}}(|\Delta^{n-1}|, Y) \quad \text{and} \\
s_i(f) &= |\Delta^{n+1}| \overset{S_i}{\twoheadrightarrow} |\Delta^n| \overset{f}{\to} Y. \in \mathrm{Hom}_{\mathsf{Top}}(|\Delta^{n+1}|, Y).
\end{aligned}
$$

Thus we see that $\mathrm{Sing}(-)$ is a functor that yields a simplicial set of the possible mappings of the standard simplex into an arbitrary topological space. The abstract idea of the face decomposition – "leaving out a vertex" – and the degeneracies – "collapsing two vertices" – becomes explicit here by precomposing with topological maps that actually do this. ⌟

Example 1.6. More generally, one can also define simplicial objects in an arbitrary category C: They are functors $X_\bullet, Y_\bullet : \Delta^{\mathsf{op}} \to \mathsf{C}$ with natural transformations as (simplicial) morphisms. For example, a *simplicial group* $G_\bullet$ is a family of groups $(G_i)_{i \in \mathbb{N}}$ with group homomorphisms $d_i : G_n \to G_{n-1}$ and $s_i : G_n \to G_{n+1}$ for all $n \in \mathbb{N}$, $i = 0, \ldots, n$, such that the relations (1^{op}) hold. Often one does not explicitly specify the functor $G_\bullet$, but rather describes how the family of groups admits the structure carried over from Δ^{op}.

Given a simplicial set $X_\bullet$, we can construct a simplicial group $C_*(X_\bullet)$ that has $C_n(X_\bullet) = \mathbb{Z}[X_n]$, the free abelian group on the generators X_n; the maps d_i and s_i are the obvious ones on the generators X_n and extend linearly. In classical algebraic topology, the chain complexes arising in singular homology are examples of simplicial abelian groups: For a topological space Y, using $X_\bullet = \mathrm{Sing}(Y)$ from the previous example yields $C_n(X_\bullet) = \mathbb{Z}[\{\sigma \colon |\Delta^n| \to Y\}]$, the singular chain complex. The "total boundary" used in homology is then just the alternating sum of the face maps: $\partial = \sum_{i=0}^n (-1)^i d_i : C_n(X_\bullet) \to C_{n-1}(X_\bullet)$. (Cf. [Fri08, Ex. 6.3]; also see [Hat02, Ch. 2] for an elementary treatment.) ⌟

Definition 1.7. The *cyclic category* Λ is obtained from Δ by adding a generating arrow $\tau_n\colon [n] \to [n]$, $n \in \mathbb{N}$, modulo the following *cocyclic relations*[1]:

$$\begin{array}{rcll} \tau_n\delta_i &=& \delta_{i-1}\tau_{n-1} \text{ for } 1 \le i \le n; & \tau_n\delta_0 = \delta_n \\ \tau_n\sigma_i &=& \sigma_{i-1}\tau_{n+1} \text{ for } 1 \le i \le n; & \tau_n\sigma_0 = \sigma_n\tau_{n+1}^2 \\ \tau_n^{n+1} &=& \mathrm{id} & \end{array} \tag{2}$$

Again, we denote with Λ^{op} the opposite category; the generating arrows are denoted by d_i, s_i and t_n with the relations (1^{op}) and (2^{op}) as follows:

$$\begin{array}{rcll} d_it_n &=& t_{n-1}d_{i-1} \text{ for } 1 \le i \le n; & d_0t_n = d_n \\ s_it_n &=& t_{n+1}s_{i-1} \text{ for } 1 \le i \le n; & s_0t_n = t_{n+1}^2 s_n \\ t_n^{n+1} &=& \mathrm{id} & \end{array} \tag{2^{op}}$$

A *cyclic set* $X_\bullet$ is a functor $X_\bullet : \Lambda^{\mathsf{op}} \to \mathsf{Set}$. The *category of cyclic sets* is the functor category $\mathrm{Func}(\Lambda^{\mathsf{op}}, \mathsf{Set})$ and is denoted by CycSet. ⌟

Remark 1.8. The cyclic category was introduced by Alain Connes in his 1983 paper *Cohomologie Cyclique et Foncteurs* Ext^n [Con83, Sect. 2]. There, the category's set of morphisms $\mathrm{Hom}_\Lambda([m],[n])$ is defined as the homotopy classes of monotone degree-1-maps $\varphi : S^1 \to S^1$ such that $\varphi(\mathbb{Z}/(m+1)\mathbb{Z}) \subseteq \varphi(\mathbb{Z}/(n+1)\mathbb{Z})$.

One can construct Δ and Λ more "explicitly" as subcategories of the category FinSet of finite sets and set-maps. Let $\delta_i : [n-1] \to [n]$ to be the unique, order-preserving, injective set-map that does not contain i in its image; $\sigma_i : [n+1] \to [n]$ to be the unique, order-preserving, surjective set-map for which the preimage $\sigma_i^{-1}(i)$ has two elements; and $\tau_n : [n] \to [n]$ via $i \mapsto i-1 \mod n+1$.

In the opposite categories these set-maps cannot be set-maps; hence the intuition behind d_i is "leave out i", and s_i means "let i occur in two successive places"; t_n is an actual set-map $i \mapsto i+1 \mod n+1$, the "rotation of corners". ⌟

Example 1.9. Analogously to example 1.4, we have the *standard n-dimensional cyclic set* $\Lambda^n(-) = \mathrm{Hom}_\Lambda(-,[n]) = \mathrm{Hom}_{\Lambda^{\mathsf{op}}}([n],-)$. Again we have $\mathrm{Hom}_{\mathsf{CycSet}}(\Lambda^n, X_\bullet) \cong X_n$ naturally.

We can endow the cosimplicial space $|\Delta^\bullet|$ with extra maps $T_n : |\Delta^n| \to |\Delta^n|$ that send the simplex corners v_i to v_{i-1}, $1 \le i \le n$, $T_n(v_0) = v_n$, and extend linearly on the faces. This turns $|\Delta^\bullet|$ into a *cocyclic space* (cf. [Lod98, 7.1.3]). ⌟

It appears like cyclic objects[2] are just simplicial objects with some extra structure. We formalize this aspect in the following

Proposition 1.10. *The following properties establish how Λ relates to Δ.*

(i) Every $f \in \mathrm{Hom}_\Lambda([m],[n])$ has a unique decomposition $f = gr$, $g \in \mathrm{Hom}_\Delta([m],[n])$, $r \in \mathrm{Aut}_\Lambda([m])$, and we have $\mathrm{Hom}_\Lambda([m],[n]) \cong \mathrm{Hom}_\Delta([m],[n]) \times C_{m+1}^{\mathsf{op}}$.

(ii) This isomorphism allows an obvious inclusion functor $j : \Delta \to \Lambda$ that induces a forgetful functor $j^* : \mathrm{Func}(\Lambda^{\mathsf{op}}, \mathsf{C}) \to \mathrm{Func}(\Delta^{\mathsf{op}}, \mathsf{C})$ that forgets the extra maps t_n.

[1] Note that the relation $\tau_n\delta_0 = \delta_n$ need not be expcicitly specified: It follows from the other relation via $\delta_n = \tau_n^{n+1}\delta_n = \tau_n^n\delta_{n-1}\tau_{n-1} = \ldots = \tau_n\delta_{n-n}\tau_{n-1}^n = \tau_n\delta_0$. Similarly, $\sigma_n\tau_{n+1}^2 = \tau_n^{n+1}\sigma_n\tau_{n+1}^2 = \tau_n^n\sigma_{n-1}\tau_{n+1}^3 = \ldots = \tau_n\sigma_{n-n}\tau_{n+1}^{2+n} = \tau_n\sigma_0$.

[2] Warning: Analogously to a "simplicial group" one could talk about "cyclic groups" in this context. However, the term is already in use, so "cyclic group" will always mean a group $G = \langle g \rangle$.

Proof. The unique decomposition in (i) follows from rules (1) and (2): Given a representation $f = f_k f_{k-1} \cdots f_1 \in \mathrm{Hom}_\Lambda([m],[n])$, $f_i \in \{\delta_i, \sigma_i, \tau^j\}$, we can w. l. o. g. assume $f_2 \neq \tau$, $f_1 = \tau^j$ for some $j \in [m]$. Let $\bar{k} = \max\{i \mid 1 \leq i \leq k, f_i = \tau\}$. If $\bar{k} > 1$, then one can exchange $f_{\bar{k}}$ with $f_{\bar{k}-1}$ using (2), and by induction this yields the desired decomposition $f = gr$. It is unique: Assume $f = gr = g'r'$, then if $r = r'$ we have $g = g'$ and are done; otherwise we have $g = g'r'r^{-1} = g'\tau^j$ for some $j \in \{1, \ldots, m\}$. For this equality to hold there must exist a relation among generators satisfying $h\tau = h'$ for some $h, h' \in \{\delta_i, \sigma_i\}$, a contradiction.

The preceding argument shows that for every choice of $f \in \mathrm{Hom}_\Delta([m],[n])$ and $j \in [m]$, we obtain a different arrow in Λ. The set $\mathrm{Aut}_\Lambda([n])$ contains exactly all τ_n^j, $j = 0, \ldots, n$, and none of the other endomorphisms (remark 1.8 is helpful to see this). We identify

$$\mathrm{Aut}_\Lambda([m]) = \{\tau_m^i \mid i = 0, \ldots, m\} \longleftrightarrow (\mathbb{Z}/(m+1)\mathbb{Z})^{\mathsf{op}} \cong C_{m+1}^{\mathsf{op}}$$

in accordance with remark 1.8, by letting τ_m be the multiplication on the right by the inverse of the generator of C_{m+1}, the cyclic group of order $m+1$. (While from an algebraic point of view the distinction between C_{m+1}^{op} and C_{m+1} is unnecessary, the notation is meant to suggest that C_{m+1} is generated by t_m, and the opposite group is generated by τ_m.) This yields the isomorphism of (i) and the dual version reads $\mathrm{Hom}_{\Lambda^{\mathsf{op}}} \cong \mathrm{Hom}_{\Delta^{\mathsf{op}}} \times C_{m+1}$.

The inclusion functor $j : \Delta \to \Lambda$ in (ii) is the identity and hence injective on the set of objects; it maps $f \mapsto f \times \mathrm{id}_{[m]} \in \mathrm{Hom}_\Lambda([m],[n])$ faithfully, so Δ is a subcategory of Λ. □

Proposition 1.11. *The category Λ is self-dual, i. e. $\Lambda \cong \Lambda^{\mathsf{op}}$.*

Note. This is not true in Δ: Already in the lowest possible dimension, the Hom-sets do not have the same cardinality, i. e. $\mathrm{Hom}_\Delta([0],[1]) = \{\delta_0, \delta_1\}$, but $\mathrm{Hom}_{\Delta^{\mathsf{op}}}([0],[1]) = \{s_0\}$. In Λ this is repaired by the fact that $\mathrm{Hom}_{\Lambda^{\mathsf{op}}}([0],[1]) = \{s_0, s_0 t_1^{-1}\}$.

Proof of Proposition 1.11. We give an explicit equivalence of the categories: It is the identity on objects, and the morphisms are mapped via $-^* : \Lambda \to \Lambda^{\mathsf{op}}$ for every $n \in \mathbb{N}$ as follows:

$$\begin{aligned}
&\delta_i^* = s_i && i = 0, \ldots, n-1 && \delta_i : [n-1] \to [n] \\
&\delta_n^* = s_0 t_n^{-1} && && \delta_n : [n-1] \to [n] \\
&\sigma_i^* = d_{i+1} && i = 0, \ldots, n-1 && \sigma_i : [n] \to [n-1] \\
&\tau_n^* = t_n^{-1} && && \tau_n : [n] \to [n]
\end{aligned} \tag{3}$$

The mapping is functorial as can be verified by checking that it is compatible with the relations (1) and (2). Exemplarily, we have

$$\begin{aligned}
s_j s_i = \delta_j^* \delta_i^* = (\delta_j \delta_i)^* = (\delta_i \delta_{j-1})^* = \delta_i^* \delta_{j-1}^* = s_i s_{j-1} \text{ for } i < j, \\
t_n^{-1} = d_1 s_0 t_n^{-1} = (\sigma_0 \delta_n)^* = (\delta_{n-1}\sigma_0)^* = s_0 t_{n-1}^{-1} d_1 = s_0 t_{n-1}^{-1} t_{n-1} d_0 t_n^{-1} = t_n^{-1}, \text{ and} \\
s_{i-1} t_{n-1}^{-1} = t_n^{-1} s_i = (\tau_n \delta_i)^* = (\delta_{i-1}\tau_{n-1})^* = s_{i-1} t_{n-1}^{-1}.
\end{aligned}$$

We have the following identities, using (2):

$$\begin{aligned}
&\delta_i^{**} = s_i^* = \delta_{i+1} = \tau_n^{-1}\tau_n\delta_{i+1} = \tau_n^{-1}\delta_i\tau_{n-1} \\
&\sigma_i^{**} = d_{i+1}^* = \sigma_{i+1} = \tau_{n-1}^{-1}\tau_{n-1}\sigma_{i+1} = \tau_{n-1}^{-1}\sigma_i\tau_n \\
&\tau_n^{**} = t_n^{*-1} = \tau_n
\end{aligned}$$

Hence for any $f \in \mathrm{Hom}_\Lambda([m],[n])$, one has $f^{**} = \tau_n^{-1} f \tau_m$. So f and f^{**} agree up to unique isomorphism, and this implies ([Awo10, 7.25]) that Λ and Λ^{op} are equivalent categories. □

Remark 1.12. The self-duality leads to an equivalent construction of Λ via an extra degeneracy $\sigma_{n+1} := \sigma_0 \tau_n^{-1}$ as used in [DHK85, p. 282]; see [Lod98, 6.1.11]. ⌟

2 Realization of Simplicial Spaces

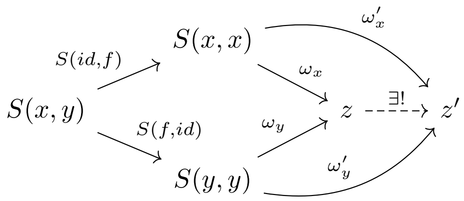
We introduce the realization functor, which will be studied carefully in the following sections. While the prototypical standard simplicial and cyclic sets are finite in every dimension, it will later be interesting to see what happens in the case of simplicial (or cyclic) *spaces*. We will generally pick as our "category of reasonably well-behaved spaces" the compactly generated weak Hausdorff spaces which form a category CGWH. (A helpful and concise list of properties can be found in [Str09].)

Definition 2.1. The *category of simplicial spaces* is the functor category $\mathrm{Func}(\Delta^{\mathsf{op}}, \mathsf{CGWH})$ and denoted by SimpSpace. The *category of cyclic spaces*, CycSpace, is defined analogously as $\mathrm{Func}(\Lambda^{\mathsf{op}}, \mathsf{CGWH})$. ⌟

Remark 2.2. The standard simplicial and cyclic *sets* regarded as simplicial and cyclic *spaces* are assumed to carry the discrete topology. This way, any simplicial (resp. cyclic) set can be regarded as a discrete simplicial (resp. cyclic) space. ⌟

The following definition will be central to defining the realization functor. It uses the general notion of the *coend* of a *dinatural transformation* developed in [ML98, IX.4–5].

Definition 2.3. Let $S : \mathsf{C}^{\mathsf{op}} \times \mathsf{C} \to \mathsf{D}$ be a functor. The *coend of* S, written $\int^{c:\mathsf{C}} S(c,c)$, is defined to be the universal dinatural transformation $\omega : S \ddot{\to} z$, $z \in \mathrm{Ob}(\mathsf{D})$, such that for any dinatural $\omega' : S \ddot{\to} z'$, there is a unique arrow completing, for all $f \in \mathrm{Hom}_{\mathsf{C}^{\mathsf{op}}}(x,y)$, the following diagram:



Given functors $F : \mathsf{C}^{\mathsf{op}} \to \mathsf{D}$ and $G : \mathsf{C} \to \mathsf{D}$ and if D has products, one also writes $F \times_{\mathsf{C}} G$ for $\int^{c:\mathsf{C}} (F \times G)(c,c)$. The definition of an *end* is dual to that of a coend, and one writes $\int_{c:\mathsf{C}} S(c,c)$. If it is clear from which category the (co)end is computed one need not indicate it, i. e. in most of the following statements, the superscript m is short for $m : \Delta$. ⌟

Since coends are just special kind of colimits, we immediately get:

Proposition 2.4. $\mathrm{Hom}(X, \int^c S) \cong \int^c \mathrm{Hom}(X, S)$ *and* $\mathrm{Hom}(\int^c S, X) \cong \int_c \mathrm{Hom}(S, X)$.

Definition 2.5. Define the *realization functor* $|-| : \mathsf{SimpSpace} \to \mathsf{CGWH}$ as follows: Given a simplicial space (or set) $X_\bullet$, set $|X_\bullet| = \int^m X_m \times |\Delta^m|$, where X_m, in case it is "just" a set, is given the discrete topology, and $|\Delta^m| \subset \mathbb{R}^{m+1}$ denotes the m-dimensional standard simplex from def. 1.1; the realization of a simplicial morphism $f_\bullet : X_\bullet \to Y_\bullet$ is defined simplex-wise. ⌟

Proposition 2.6. *Let $X_\bullet$ be a simplicial space.*

(i) The realization can be explicitly computed as

$$|X_\bullet| = \coprod_{n=0}^{\infty} X_n \times |\Delta^n| \Big/ \sim,$$

with $(x, D_i(p)) \sim (d_i x, p)$ and $(x, S_i(p)) \sim (s_i x, p)$.

(ii) The space $|X_\bullet|$ is compactly generated and weak Hausdorff.

(iii) Given a simplicial map $f_\bullet : X_\bullet \to Y_\bullet$, $|f_\bullet|([x,p]) = [f_n(x), p]$ is well-defined and continuous.

(iv) The realization of Δ^n is indeed the standard simplex $|\Delta^n|$, justifying our previous notation.

Proof. For (i), we can rephrase definition 2.3: The coend of $S(m,n) = X_m \times |\Delta^n|$ is the coequalizer

$$\operatorname{coequ}\left(\coprod_{s \in \operatorname{Mor}\Delta} X_{\operatorname{cod} s} \times |\Delta^{\operatorname{dom} s}| \underset{s^* \times \mathrm{id}}{\overset{\mathrm{id} \times s_*}{\rightrightarrows}} \coprod_{n \in \operatorname{Ob}(\Delta)} X_n \times |\Delta^n| \right),$$

hence we "glue together" one copy of the standard n-simplex along X_n in every dimension $n \in \mathbb{N}$ in precisely the way the cells relate to each other in $X_\bullet$. Explicitly, we identify $(s_i(x), p) \in (X_{n+1}, |\Delta^{n+1}|)$ with $(x, S_i(p)) \in (X_n, |\Delta^n|)$, and $(d_i(x), p) \in (X_n, |\Delta^n|)$ with $(x, D_i(p)) \in (X_{n+1}, |\Delta^{n+1}|)$.

To prove (ii), we observe that gluing together CGWH-spaces in this fashion yields a CGWH-space again: We assume X_n to be CG (and WH), and since $|\Delta^n|$ is a (locally) compact Hausdorff space, $X_n \times |\Delta^n|$ is CG (and WH) [Str09, prop. 2.6]. It follows that $\coprod_n X_n \times |\Delta^n|$ is CG (and WH) by [Str09, prop. 2.2]. Modding out by $\sim$ retains the CG property [Str09, prop. 2.1]; but more importantly it also preserves the WH property: the relation $\sim$ is obtained via continuous functions, and combining [Str09, cor. 2.15] and [Str09, prop. 2.22] guarantees that $|X_\bullet|$ seen as the colimit computed in Top agrees with the colimit in CGWH.

(iii): A simplicial morphism $f_\bullet : X_\bullet \to Y_\bullet$ is a natural transformation of the involved functors $X_\bullet$ and $Y_\bullet$. The realization of $f_\bullet$ is denoted by $|f_\bullet| : |X_\bullet| \to |Y_\bullet|$ and is defined "simplex-wise", i. e. for every $n \in \mathbb{N}$, $x \in X_n$ we have

$$|f_\bullet|\big|_{(x, |\Delta^n|)} : \{x\} \times |\Delta^n| \to |Y_\bullet|, \quad (x, p) \mapsto (f_n(x), p) \in Y_n \times |\Delta^n|.$$

The naturality of $f_\bullet$ means there are commutative diagrams of the form

$$\begin{array}{ccccc} X_{n+1} & \xleftarrow{s_i} & X_n & \xrightarrow{d_i} & X_{n-1} \\ \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} \\ Y_{n+1} & \xleftarrow{s_i} & Y_n & \xrightarrow{d_i} & Y_{n-1} \end{array}$$

and thus the map $|f_\bullet|$ is compatible with the relation $\sim$, because

$$\begin{aligned} |f_\bullet|(x, D_i(p)) &= (f_n(x), D_i(p)) = (d_i \circ f_n(x), p) \\ &= (f_{n-1} \circ d_i(x), p) = |f_\bullet|(d_i x, p), \quad \text{and} \\ |f_\bullet|(x, S_i(p)) &= (f_n(x), S_i(p)) = (s_i \circ f_n(x), p) \\ &= (f_{n+1} \circ s_i(x), p) = |f_\bullet|(s_i x, p). \end{aligned}$$

(iv): It is clear that any simplex assigned to a degenerate cell "vanishes" in the realization, as it will be identified with a face of the simplex of a nondegenerate cell; in the same way, faces obtained from nondegenerate cells will be glued on to the corresponding simplex. So it is clear that our notation of $|\Delta^n|$ is justified: it is enough to see that Δ^n has only one nondegenerate cell in dimensions $\geq n$, namely $\mathrm{id}_{[n]} \in \Delta^n[n] = \operatorname{Hom}_\Delta([n],[n])$, and all nondegenerate cells in dimensions $< n$ can be obtained as faces. ☐

Proposition 2.7. *A simplicial space $X_\bullet$ has a canonical presentation $X_\bullet \cong \int^m X_m \times \Delta^m[\bullet]$.*

Proof. For any $n \in \mathbb{N}$, the space $\Delta^m[n] = \operatorname{Hom}_{\Delta^{\mathrm{op}}}([m],[n])$ carries the discrete topology, and $X_m \times \Delta^m[n]$ inherits the product topology. Analogously to the proof of 2.6, we find that X_n is homeomorphic to the quotient

$$Y_n = \coprod_{m=0}^{\infty} X_m \times \Delta^m[n] \Big/ (X(f) \sim \Delta^f[n]).$$

Certainly the maps $X_n \to Y_n, x \mapsto [x, \mathrm{id}_{[n]}]$ and $Y_n \to X_n, [y, (s\colon [m] \to [n])] \mapsto X(s)(y)$ are well-defined, continuous and compose to give the identity map on X_n resp. Y_n, hence the spaces are homeomorphic. By construction, the level-wise homeomorphisms are natural in n and thus assemble to a simplicial isomorphism. □

We will conclude this section with a classic result about simplicial sets: Giving a continuous map from $|X_\bullet|$ to a topological space Y is "essentially the same" as giving a simplicial morphism $X_\bullet \to \mathrm{Sing}(Y)$, where $\mathrm{Sing}(Y) = \mathrm{Hom}(|\Delta^\bullet|, Y)$ is the simplicial set introduced in example 1.5.

Theorem 2.8. *Realization of a simplicial set is left adjoint to* Sing, *i. e.*

$$\mathrm{Hom}_{\mathsf{Top}}(|X_\bullet|, Y) \cong \mathrm{Hom}_{\mathsf{S}}(X_\bullet, \mathrm{Sing}(Y)).$$

Proof. Let $X_\bullet$ be a simplicial set (or discrete space), and Y a topological space. We have $X_\bullet \cong \int^m X_m \times \Delta^m$ naturally by proposition 2.7. Using prop. 2.4, we formally compute:

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{S}}(X_\bullet, \mathrm{Sing}(Y)) &\cong \mathrm{Hom}_{\mathsf{S}}\left(\int^m X_m \times \Delta^m,\ \mathrm{Sing}(Y)\right) \\
&\cong \int_m \mathrm{Hom}_{\mathsf{S}}(X_m \times \Delta^m, \mathrm{Sing}(Y)) \\
&\cong \int_m \mathrm{Hom}_{\mathsf{Set}}(X_m, \mathrm{Hom}_{\mathsf{S}}(\Delta^m, \mathrm{Sing}(Y))) \\
&\overset{(*)}{\cong} \int_m \mathrm{Hom}_{\mathsf{Set}}(X_m, \mathrm{Hom}_{\mathsf{Top}}(|\Delta^m|, Y)) \\
&\cong \int_m \mathrm{Hom}_{\mathsf{Top}}(X_m \times |\Delta^m|, Y) \\
&\cong \mathrm{Hom}_{\mathsf{Top}}\left(\int^m X_m \times |\Delta^m|,\ Y\right) \\
&= \mathrm{Hom}_{\mathsf{Top}}(|X_\bullet|, Y)
\end{aligned}$$

The isomorphism $(*)$ follows from the Yoneda lemma and characterizes the standard simplicial set Δ^m (see example 1.4): $\mathrm{Hom}_{\mathsf{S}}(\Delta^m, \mathrm{Sing}(Y)) \cong \mathrm{Sing}(Y)_m = \mathrm{Hom}_{\mathsf{Top}}(|\Delta^m|, Y)$. All isomorphisms are natural. □

Remark 2.9. The above proof hinges on the Yoneda lemma, a statement about Hom-*sets*, and the existence of an adjoint to the cartesian product in the category Set. The latter has an analogue in the cartesian closed category CGWH, and $\mathrm{Sing}(Y)$ for a CGWH-space Y could be level-wise topologized using the compact-open topology to form a simplicial space. Enriching Hom-sets with additional structure (such as a topology) and proving a Yoneda-type result for functors $\mathsf{C}^{\mathsf{op}} \to \mathsf{D}$ is possible [Kel05, section 2.4]; it is, however, beyond the scope of this text to carefully check that this yields indeed an adjunction between realization of simplicial *spaces* and the singular simplicial *space* functor. ⌟

3 The Realization Functor is Finitely Continuous

We describe finite products and equalizers in the category of simplicial spaces and prove that the realization functor preserves all finite limits.

Proposition 3.1. *Let $X_\bullet$ and $Y_\bullet$ be simplicial spaces. Then the product $X_\bullet \times Y_\bullet$ exists and is given level-wise by $(X_\bullet \times Y_\bullet)_n = X_n \times Y_n$, with structure-maps component-wise, i. e. $f(x, y) = (f(x), f(y))$.*

Proof. First note that the projections $X_\bullet \times Y_\bullet \to X_\bullet$ and $X_\bullet \times Y_\bullet \to Y_\bullet$ are simplicial, then note that $X_\bullet \times Y_\bullet$ is universal with respect to this property: Given a simplicial space $Z_\bullet$ and simplicial maps $p_\bullet : Z_\bullet \to X_\bullet$ and $q_\bullet : Z_\bullet \to Y_\bullet$, we have a unique level-wise continuous map $(p_\bullet, q_\bullet) : Z_\bullet \to X_\bullet \times Y_\bullet$ by universality of the product in CGWH, and this map is simplicial. ☐

Proposition 3.2. *There exists a continuous bijection between the sets $|X_\bullet \times Y_\bullet|$ and $|X_\bullet| \times |Y_\bullet|$.*

Proof. Let $P : |X_\bullet \times Y_\bullet| \to |X_\bullet| \times |Y_\bullet|$ be defined via $[(x,y),p] \mapsto ([x,p],[y,p])$. This map is obviously well-defined, and allows for an inverse map $P^{-1} : |X_\bullet| \times |Y_\bullet| \to |X_\bullet \times Y_\bullet|$ as follows: Given a point $([x,p],[y,q])$ with x, y nondegenerate and $p = (p_0, \dots, p_m) \in |\Delta^m|$, $q = (q_0, \dots, q_n) \in |\Delta^n|$, consider the (totally ordered) set

$$S := \{\, p_{\le j} \,|\, j \le m\} \cup \{\, q_{\le l} \,|\, l \le n\} \;\text{ for } p_{\le j} = \sum_{i=0}^{j} p_i \text{ and } q_{\le l} = \sum_{k=0}^{l} p_k$$

and create from it the "tuple of differences"

$$s = (s_0, s_1 - s_0, s_2 - s_1, \dots, s_{|S|} - s_{|S|-1}) \in |\Delta^{|S|+1}| \quad (s_i < s_j \forall i < j).$$

It is clear by construction that we have

$$S_{i_0} \cdots S_{i_k}(s) = p \quad \text{ and } \quad S_{j_0} \cdots S_{j_l}(s) = q$$

for a unique choice of $i_0 < \cdots < i_k$ and $j_0 < \cdots < j_l$ (each application collapsing two adjacent coordinates by summing them up). Thus we define the map P^{-1} via

$$([x,p],[y,q]) = ([s_{i_k} \cdots s_{i_0}(x), s], [s_{j_l} \cdots s_{j_0}(y), s]) \mapsto [(s_{i_k} \cdots s_{i_0}(x), s_{j_l} \cdots s_{j_0}(y)), s].$$

It is clear that $PP^{-1} = \mathrm{id} = P^{-1}P$, and P is also continuous: By functoriality of $|{-}|$ the realization of the simplicial projections are continuous, thus $|\operatorname{proj}_1| = \operatorname{proj}_1 \circ P$ and $|\operatorname{proj}_2| = \operatorname{proj}_2 \circ P$ are continuous and by universality of the product ([Str09, prop. 2.4]) P is continuous. ☐

Corollary 3.3. *We have a homeomorphism $|\Delta^m \times \Delta^n| \cong |\Delta^m| \times |\Delta^n|$.*

Proof. The domain is *compact* and the codomain is a Hausdorff space. Thus we employ a standerd argument to show that the bijection P being continuous already implies it is a homeomorphism. Let $u : K \to H$ be a continuous map from a compact space K to a Hausdorff space H, and let $C \subset K$ be closed. This implies C is compact, thus $u(C)$ compact, and since H is Hausdorff, $u(C)$ is closed. Thus u is a closed continuous bijection, hence a homeomorphism. ☐

The general case now follows:

Theorem 3.4. *For simplicial CGWH-spaces $X_\bullet$ and $Y_\bullet$, the continuous bijection $P : |X_\bullet \times Y_\bullet| \to |X_\bullet| \times |Y_\bullet|$ is a homeomorphism.*

Proof. Note that CGWH is cartesian closed [Str09, 2.24], so the cartesian product has a right-adjoint and thus preserves colimits in both factors. We formally compute:

$$\begin{aligned}
|X_\bullet \times Y_\bullet| &\cong \left| \left(\int^m X_m \times \Delta^m \right) \times \left(\int^n Y_n \times \Delta^n \right) \right| \\
&\cong \left| \int^m \int^n X_m \times Y_n \times \Delta^m \times \Delta^n \right| \\
&\cong \int^m \int^n X_m \times Y_n \times |\Delta^m \times \Delta^n| \\
&\cong \int^m \int^n X_m \times Y_n \times |\Delta^m| \times |\Delta^n|
\end{aligned}$$

$$\cong \left(\int^m X_m \times |\Delta^m|\right) \times \left(\int^n Y_n \times |\Delta^n|\right)$$
$$\cong |X_\bullet| \times |Y_\bullet|$$

□

Proposition 3.5. *The category* SimpSpace *has all equalizers of pairs of arrows, and the (simplicial) equalizer* $E_\bullet \to X_\bullet \rightrightarrows Y_\bullet$ *agrees level-wise with the one computed in* CGWH*.*

Proof. Let $X_\bullet$ and $Y_\bullet$ be simplicial spaces, and $f_\bullet, g_\bullet : X_\bullet \to Y_\bullet$ simplicial maps. Then for $n \in \mathbb{N}$, the equalizer of $X_n \rightrightarrows Y_n$ is the space $E_n = \{x \in X_n \mid f_n(x) = g_n(x)\}$ topologized as a subspace of X_n, and $e_n : E_n \hookrightarrow X_n$ is a closed inclusion [Str09, 3.1(b)]. Because $f_\bullet$ and $g_\bullet$ are by assumption simplicial and continuous, so is $e_\bullet$. Thus SimpSpace has all equalizers of pairs of arrows. □

Proposition 3.6. *The realization functor preserves equalizers.*

Proof. In the above setup, denote the equalizer in CGWH of $|f_\bullet|, |g_\bullet| : |X_\bullet| \to |Y_\bullet|$ by E, and denote by e the closed inclusion $e : E \hookrightarrow |X_\bullet|$. Since

$$E = \left\{ [\sigma, p] \in \coprod X_n \times |\Delta^n| \big/ \sim \;\middle|\; |f_\bullet|([\sigma, p]) = |g_\bullet|([\sigma, p]) \right\}$$
$$= \{[\sigma, p] | [f_n\sigma, p] = [g_n\sigma, p]\} = |E_\bullet|,$$

the spaces E and $|E_\bullet|$ agree *as sets* and $|e_\bullet| = e$. By proving that $|e_\bullet|$ is a closed inclusion, we establish that both spaces are topologized equivalently. We know from the previous proposition that $e_n : E_n \hookrightarrow X_n$ is a closed inclusion for all n, and by [Str09, 2.32], the map

$$\coprod E_n \times |\Delta^n| \hookrightarrow \coprod X_n \times |\Delta^n|$$

is a closed inclusion as well. We have the obvious quotient maps onto $|E_\bullet|$ and $|X_\bullet|$, and can now chase around open sets directly: A subset $U \subset |E_\bullet|$ is open if and only if there exists V open in $\coprod E_n \times |\Delta^n|$ such that $V/\sim = U$, if and only if there exists W open in $\coprod X_n \times |\Delta^n|$ such that $V = W \cap \coprod E_n \times |\Delta^n|$. Projecting W onto the open set $W/\sim \subset |X_\bullet|$, we find that $W/\sim \cap |E_\bullet| = (W \cap \coprod E_n \times |\Delta^n|)/\sim = V/\sim = U$, so $U \subset |E_\bullet|$ is open if and only it is open in $|E_\bullet| \subset |X_\bullet|$. Thus $|e_\bullet|$ is a closed inclusion, and therefore both E and $|E_\bullet|$ carry the subspace topology. □

Remark 3.7. Given a cyclic space, we will later be interested in the space of fixed points under a certain group-action. Let us for now remark that if a space $X_\bullet$ comes with a level-wise simplicial G-action, for G a finite cyclic group, then the space of fixed points under this action is the equalizer of the identity map and multiplication by the generator g of G, denoted by μ_g. By the preceding statement we find:

$$|X_\bullet^G| = |\{x \mid g \cdot x = x\}| = \left|\mathrm{eq}\left(X_\bullet \overset{\mathrm{id}_\bullet}{\underset{\mu_g}{\rightrightarrows}} X_\bullet \right)\right| \cong \mathrm{eq}\left(|X_\bullet| \overset{\mathrm{id}}{\underset{|\mu_g|}{\rightrightarrows}} |X_\bullet| \right) = |X_\bullet|^G,$$

where the action of G on $|X_\bullet|$ is understood to be induced by μ_g. ⌟

We quickly collect evident statements about terminal objects in both categories:

Remark 3.8. The category CGWH has a terminal object, the one-point space $\{*\}$. The category SimpSpace has a terminal object, namely the terminal object of CGWH in every dimension n. The realization functor preserves the terminal object. ⌟

We wrap up the section with a generalized statement about the preservation of limits:

Theorem 3.9. *The realization functor is finitely continuous, i. e. it preserves all finite limits.*

Proof. The limit of the empty diagram is preserved by the previous remark. It is a well-known category-theoretical fact that if a category has a terminal object, all binary products and all equalizers of pairs of arrows, then it has all finite limits (and is called *finitely complete*); in case the indexing diagram is not empty, this limit can be explicitly constructed as the equalizer of finite products as outlined in [ML98, V.2 Thm. 1]. Since realization preserves both binary (and thus by induction finite) products and equalizers, it preserves all finite limits. □

4 Realization, Cofibrations and Weak Equivalences

This section is devoted to studying the realization functor, and how it behaves with respect to cofibrations and weak equivalences. Hence we are mainly concerned with simplicial *spaces* and not just simplicial sets (which are equivalent to discrete spaces, and not very interesting for this discussion).

Specifically, given a map $f_\bullet : X_\bullet \to Y_\bullet$ of simplicial spaces, we are interested in the following two questions:

1. If $f_\bullet$ is a level-wise weak equivalence, is $|f_\bullet|$ a weak equivalence, too?
2. If $f_\bullet$ is a level-wise cofibration, is $|f_\bullet|$ a cofibration, too?

For the convenience of the reader, we will quickly recall two standard definitions tuned to our setup:

Definition 4.1. A continuous map $f_\bullet : X_\bullet \to Y_\bullet$ is a *levelwise weak equivalence* if for all $n \in \mathbb{N}$ it induces isomorphisms on all homotopy groups, i. e. for all $n \in \mathbb{N}$ and base points $x_0 \in X_n$, the induced map $\pi_i(X_n, x_0) \xrightarrow{\cong} \pi_i(Y_n, f_n(x_0))$ is an isomorphism for all $i \in \mathbb{N}$.

The map is a *levelwise cofibration* if f_n is a cofibration for all $n \in \mathbb{N}$. A *(Hurewicz-)cofibration* must satisfy the *homotopy extension property* (HEP) which states that for all spaces Z, initial conditions $c : Y_n \to Z$ and homotopies $h : X_n \to Z^I$, we can extend h to a homotopy $H : Y_n \to Z^I$ with $\mathrm{ev}_0 \circ H = c$. Put differently, the dashed arrow in the following diagram must exist:

$$\begin{array}{ccc} X_n & \xrightarrow{\;h\;} & Z^I \\ \Big\downarrow{\scriptstyle f_n} & \overset{H}{\nearrow} & \Big\downarrow{\scriptstyle ev_0} \\ Y_n & \xrightarrow{\;c\;} & Z \end{array}$$

⌟

It turns out that without some "goodness" condition on the spaces the realization functor will not send level-wise weak equivalences (resp. cofibrations) to weak equivalences (resp. cofibrations). The relation $d_i s_i = \mathrm{id}$ from (1^{op}) already implies that all degeneracies s_i are closed inclusions (cf. [Str09, 2.29]), but we need a slightly stronger property:

Definition 4.2. A simplicial space $X_\bullet$ is called *good* if all degeneracies are cofibrations. ⌟

To convince the reader that some goodness assumptions on the spaces $X_\bullet$ and $Y_\bullet$ are needed at all, we give a counterexample[3] where one space has a degeneracy map that is *not* a cofibration.

[3] The idea is due to Tyler Lawson as described in `http://mathoverflow.net/a/171423`.

Example 4.3. Denote by N the subspace $\{0\}\cup\{\frac{1}{2},\frac{1}{3},\frac{1}{4},\dots\}\subset\mathbb{R}$, and by N_d the same set but this time with the discrete topology. The bijection $N_d\to N$ is continuous but not a homeomorphism. While $\{0\}$ includes into N_d as a cofibration, the map $\{0\}\hookrightarrow N$ is the standard example of a closed inclusion that is *not* a cofibration.

Define $X_0=Y_0=\{0\}$, and denoting by 0 the base-point of N and N_d, set $X_n=\bigvee_{i=1}^n N_d$ and $Y_n=\bigvee_{i=1}^n N$. The face and degeneracy maps of $Y_\bullet$ are given as follows, and for $X_\bullet$ they are defined completely analogously:

$$d_i:\bigvee_{j=1}^{n}N_{(j)}\to\bigvee_{j=1}^{n-1}N_{(j)},\quad x\mapsto\begin{cases}0, & x\in N_{(1)}\text{ and }i=0\\ x\in N_{(k-1)}, & x\in N_{(k)},\ 0\le i\le k\\ x\in N_{(k)}, & x\in N_{(k)},\ i>k\end{cases}$$
$$s_i:\bigvee_{j=1}^{n-1}N_{(j)}\to\bigvee_{j=1}^{n}N_{(j)},\quad x\mapsto\begin{cases}x\in N_{(k)}, & x\in N_{(k)}\text{ for }k<i\\ x\in N_{(k+1)}, & x\in N_{(k)}\text{ for }k\ge i\end{cases}$$

The d_i and s_i are certainly continuous for $X_\bullet$ and $Y_\bullet$, and the simplicial relations are satisfied, so we have two simplicial spaces. But since the degeneracy $Y(s_0):Y_0\to Y_1$ is not a cofibration, the map $\mathrm{id}_\bullet:X_\bullet\to Y_\bullet$ is *not* a map of *good* simplicial spaces.

Since both N and N_d are totally disconnected, we have $\pi_0(X_n,x_0)\cong\pi_0(Y_n,x_0)\cong\mathbb{N}$ for all $n\ge1$ and basepoints x_0, and all homotopy groups in dimension $n\ge1$ are trivial, so $\mathrm{id}_\bullet$ is a level-wise weak equivalence. But the realization $|\mathrm{id}_\bullet|:|X_\bullet|\to|Y_\bullet|$ is not a weak equivalence as will be shown.

Both simplicial spaces feature the same underlying sets, and have a single 0-cell. The 1-cells are given by the spaces N and N_d, respectively (and it is important here that they differ, because both encode a different topological relation between the simplices). All higher-dimensional cells are degenerate, hence don't influence the realization. The realization of Δ^1 is homeomorphic to the unit interval $I=[0;1]\subset\mathbb{R}$, and we have

$$|X_\bullet|\cong(\{*\}\sqcup N_d\times I)/\{*\}\sim(N_d\times\{0,1\}\cup\{0\}\times I)\quad\text{and}$$
$$|Y_\bullet|\cong(\{*\}\sqcup N\times I)/\{*\}\sim(N\times\{0,1\}\cup\{0\}\times I)$$

The realizations of $X_\bullet$ and $Y_\bullet$ are the same sets, but carry different topological information: Since N_d is discrete, any open set in $\{x\}\times(I-\{0,1\})$, is an open set in $|X_\bullet|$, too. Therefore the realization of $X_\bullet$ is homeomorphic to a (countable) wedge of S^1's. This implies its fundamental group is freely generated on a countable set of generators.

On the other hand, $|Y_\bullet|$ is homeomorphic to the Hawaiian Earring, i. e. the space

$$H=\bigcup_{n=1}^{\infty}\left\{z\in\mathbb{C}\,\middle|\,\left|z-\frac{1}{n}\right|=\frac{1}{n}\right\}\subset\mathbb{C}.$$

This homeomorphism can be understood by "bending" the ends of $N\times I$, which are identified to a point $*$, towards $0\in\mathbb{C}$; a compact-to-Hausdorff argument confirms that this map is indeed a homeomorphism. The fundamental group of the Hawaiian Earring is not even free [DS92], thus $|\mathrm{id}_\bullet|$ cannot induce an isomorphism on the fundamental groups and therefore it is not a weak equivalence. See figure 1 for a graphical representation that highlights the topological differences between the two realizations. ⌟

We now prepare statements for the case where we have a map between two good simplicial spaces. The following definitions and observations are adapted from Waldhausen [Wal85, 1.1–2]:

Definition 4.4. A *category with cofibrations* is a category C together with a subcategory $\operatorname{co}\mathsf{C}$, whose morphisms are called *cofibrations* and are denoted by "arrows with tails" $\bullet\rightarrowtail\bullet$. The cofibration subcategory must satisfy the following three axioms:

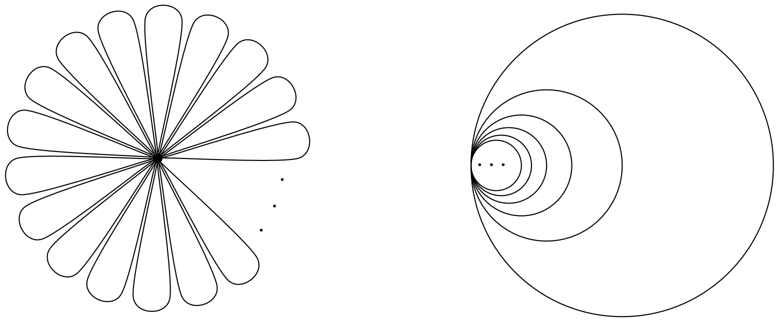


Figure 1: The realization of $X_\bullet$ on the left is a countable wedge of circles, and each circle without basepoint 0 is an open set. The realization of $Y_\bullet$ on the right differs insofar as N carries a different topology than the discrete N_d, and thus the realized 1-simplices, which are the circles, are "near to one another" in a neighborhood of 0. The topology of H is coarser and admits "more" continuous maps $S^1 \to H$, including loops that go around an infinite number of circles (see [DS92] for a detailed construction). This makes a difference for the fundamental group functor $\pi_1(-)$, and $|X_\bullet|$ and $|Y_\bullet|$ cannot be weakly equivalent.

Cof. 1 Isomorphisms of C are cofibrations,

Cof. 2 The inclusion of the initial object is a cofibration and

Cof. 3 Cofibrations admit cobase change.

Similarly, a *category with weak equivalences* has a subcategory with weak equivalence morphisms satisfying

Weq. 1 Isomorphisms of C are weak equivalences and

Weq. 2 The gluing lemma (see 4.9) holds.

Weak equivalences are denoted by $\bullet \xrightarrow{\sim} \bullet$. ⌟

Proposition 4.5. *The category of compactly generated weak Hausdorff spaces* CGWH *with the usual notion of cofibration (via the homotopy extenision property, HEP, for all spaces) and weak equivalence (via induced isomorphisms on homotopy groups) is a category with cofibrations and weak equivalences.*

Proof. Homeomorphisms are both cofibrations and weak equivalences. The initial object in unpointed CGWH-spaces is the empty space $\varnothing$, which trivially includes as a cofibration into any space Z. Cofibrations admit cobase change [Die08, Prop. 5.1.8].

A corresponding statement about weak equivalences is also true: The pushout of a weak equivalence along a cofibration is again a weak equivalence; see the following lemma 4.6. This property is also known as *left-properness*, and is required for (and, indeed, equivalent to) the gluing lemma, which is proved further down in lemma 4.9. Let us at this point remark that the *saturation axiom* [Wal85, p. 327] for weak equivalences holds in CGWH: For any two composable maps a and b, if any two of the three possibilities a, b, and $a \circ b$ are weak equivalences, then so is the third. This *two-out-of-three* argument is immediate if one thinks of the induced triangle in homotopy groups where two of the three maps are isomorphisms. □

Lemma 4.6 (Left-Properness)**.** *The category* CGWH *is left-proper, i. e. the pushout of a weak equivalence along a (Hurewicz-)cofibration yields again a weak equivalence.*

Proof. Given a pushout square

$$\begin{array}{ccc} A & \xrightarrow[\sim]{f} & B \\ {\scriptstyle i}\downarrow & \ulcorner & \downarrow{\scriptstyle I} \\ C & \xrightarrow{F} & P \end{array}$$

with i (and thus by cobase change I) a cofibration, and f a weak equivalence, we need to show that F is a weak equivalence. The stronger assumption that f is a *homotopy* equivalence implies that F is a homotopy equivalence (cf. [Die08, 5.1.10]) and therefore a weak equivalence.

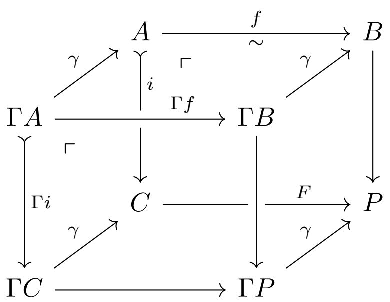
We can without loss of generality reduce the problem to the class of squares where f is a weak equivalence and a cofibration simultaneously (such a map is commonly referred to as an *acyclic* cofibration). Any continuous map factors as a cofibration followed by a homotopy equivalence [Die08, 5.3]. We can thus factor f as $h \circ j$ as follows, where $M(f) = A \times I \cup_f B$ denotes the mapping cylinder:

$$\begin{array}{ccccc} A & \overset{j}{\rightarrowtail} & M(f) & \xrightarrow[\text{h-eq}]{h} & B \\ {\scriptstyle i}\downarrow & \ulcorner & \downarrow{\scriptstyle I} & \ulcorner & \downarrow \\ C & \overset{J}{\rightarrowtail} & P' & \xrightarrow[\text{h-eq}]{H} & P \end{array}$$

Note that since the inner left hand square and the outer rectangle are both pushouts, so is the inner right hand square ("pasting of pushout diagrams"). By cobase change, I is a cofibration, and since h is a homotopy equivalence, H is one, too. Two-out-of-three yields that j is a weak equivalence (and a cofibration). We will show via CW-approximation that J is a weak equivalence, too, so the composition $H \circ J = F$ is a weak equivalence.

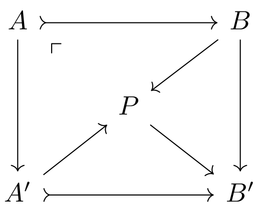
With this argument we can reduce to the case that f is a cofibration. We use the CW-approximation functor Γ from [May99, Ch. 10, sec. 5–7] and the machinery subsequently developed there. First note that a pair of spaces (A, C) has a CW-approximation $(\Gamma A, \Gamma C)$ with ΓA a subcomplex of ΓC. Subcomplexes include as cofibrations [Die08, 8.3.9], so Γ preserves cofibrations.

Because both i and f are cofibrations, we can regard A as $B \cap C$ and P as $B \cup C$. In general, the triad $(P; B, C)$ will not be an *excisive* triad (which needs $P = \mathring{B} \cup \mathring{C}$); however, with the "simple, but important general construction" around the lemma in [May99, p. 80], this triad is homotopy-equivalent to an excisive triad (also cf. [Die08, 5.3.4], which is the gluing lemma for homotopy equivalences). Hence we can apply the CW-approximation for the triad $(P; B, C)$ because it is w. l. o. g. excisive. This yields a cube diagram with diagonal maps γ that are weak equivalences:



By two-out-of-three, the assumption that f is a weak equivalence implies that Γf is a weak equivalence. The Whitehead theorem [May99, p. 76] implies that this weak equivalence of CW-complexes is in fact a homotopy equivalence. Pushing out Γf along the cofibration Γi yields a homotopy equivalence $\Gamma C \to \Gamma P$, which is also a weak equivalence. A two-out-of-three argument shows that F is a weak equivalence, finishing the proof. $\square$

Definition 4.7. Let C be a category with cofibrations and weak equivalences, and consider a square $(A \rightarrowtail B) \to (A' \rightarrowtail B')$. By forming the pushout P of $A' \leftarrow A \rightarrowtail B$, we obtain a diagram of the form



(4)

where the map $P = A' \cup_A B \to B'$ is uniquely determined by the pushout's universal property. If the maps $A \rightarrowtail A'$ and $P \rightarrowtail B'$ are cofibrations, the square (4) is called a *Waldhausen cofibration square*. If $A \xrightarrow{\sim} A'$ and $P \xrightarrow{\sim} B'$ are weak equivalences, (4) is called a *Waldhausen weak equivalence square.* ⌟

Lemma 4.8. *We state three fairly simple, but important observations about diagrams of this shape in* CGWH*:*

(i) Given a Waldhausen cofibration square, all maps in it are cofibrations, in particular $B \rightarrowtail B'$. In a Waldhausen weak equivalence square, $B \xrightarrow{\sim} B'$ is a weak equivalence.

(ii) If $A \rightarrowtail A'$ and $B \rightarrowtail B'$ in (4) are cofibrations, and $A = A' \cap B \subset B'$, the square is a Waldhausen cofibration square and in particular $P \rightarrowtail B'$ is a cofibration.

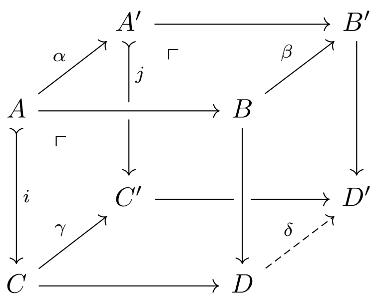
(iii) If $A \xrightarrow{\sim} A'$ and $B \xrightarrow{\sim} B'$ are weak equivalences, the square (4) is a Waldhausen weak equivalence square and in particular $P \xrightarrow{\sim} B'$ is a weak equivalence.

Proof. (i) Cobase change yields a cofibration $B \rightarrowtail P$, and composition with $P \rightarrowtail B'$ yields a cofibration $B \rightarrowtail B'$. Analogously, a weak equivalence $A \xrightarrow{\sim} A'$ determines by left-properness a weak equivalence $B \xrightarrow{\sim} P$, and composition with $P \xrightarrow{\sim} B'$ again yields weak equivalence.

(ii) Note that $A = A' \cap B$ includes into B' as a cofibration. The only map to check to obtain the claim is the induced map $P \to B'$, uniquely determined by $A' \rightarrowtail B'$ and $B \rightarrowtail B'$. In CGWH, all cofibrations are *closed* inclusions ([Str09, Cor. 2.29] in conjunction with [Die08, 5.1.2]). It follows by Lillig's cofibration union theorem (repeated in [Die08, 5.4.5]), that this map, which is the union of two closed cofibrations, is again a cofibration.

(iii) By left-properness, $B \xrightarrow{\sim} P$ is a weak equivalence, and $B \xrightarrow{\sim} B'$ is one by assumption. By two-out-of-three, $P \xrightarrow{\sim} B'$ is a weak equivalence. □

Lemma 4.9 (Gluing Lemma)**.** *Given a cube diagram in* CGWH *with front and back face pushouts, i and j cofibrations, and weak equivalences α, β and γ:*



Then the map δ is a (uniquely determined) weak equivalence.

Proof. Similarly to the proof of lemma 4.6, we "factor" the cube's top face by factoring the map $A \to B$ as a cofibration into the mapping cylinder MA followed by a homotopy equivalence. In the following diagram the left square is a pushout, by left-properness the map $MA \xrightarrow{\sim} P$ is a weak equivalence and the map $P \xrightarrow{\sim} B'$, which is uniquely determined, is therefore a weak equivalence by two-out-of-three:

$$\begin{array}{ccccc}
A & \rightarrowtail & MA & \xrightarrow{\sim} & B \\
{\scriptstyle\sim}\downarrow{\scriptstyle\alpha} & \ulcorner & \downarrow{\scriptstyle\sim} & & {\scriptstyle\sim}\downarrow{\scriptstyle\beta} \\
A' & \rightarrowtail & P & \underset{!}{\xrightarrow{\sim}} & B'
\end{array}$$

Thus it is enough to prove the lemma under the additional assumption that both top horizontal maps are a) cofibrations or b) weak equivalences. The latter is straightforward: If $A \to B$ and $A' \to B'$ are weak equivalences, immediately $C \xrightarrow{\sim} B \cup_A C$ and $C' \xrightarrow{\sim} B' \cup_{A'} C'$. Together with the given weak equivalence $\gamma : C \xrightarrow{\sim} C'$ and a two-out-of-three argument we find that δ is a weak equivalence.

Assuming the maps $A \to B$ and $A' \to B'$ are cofibrations, we can construct the desired weak equivalence by composing two weak equivalences obtained in the following fashion:

$$\begin{array}{ccccc}
A & \rightarrowtail & C & \xrightarrow{\sim} & C' \\
\downarrow & \ulcorner & \downarrow & \ulcorner & \downarrow \\
B & \rightarrowtail & B \cup_A C & \underset{(1)}{\xrightarrow{\sim}} & B \cup_A C'
\end{array}
\quad \text{and} \quad
\begin{array}{ccccc}
A & \xrightarrow{\sim} & A' & \rightarrowtail & C' \\
\downarrow & \ulcorner & \downarrow & \ulcorner & \downarrow \\
B & \xrightarrow{\sim} & B \cup_A A' & \rightarrowtail & C' \cup_A B \\
 & \searrow^{\sim} & {\scriptstyle !}\downarrow{\scriptstyle\sim} & \ulcorner & {\scriptstyle (2)}\downarrow{\scriptstyle\sim} \\
 & & B' & \rightarrowtail & B' \cup_{A'} C'
\end{array}$$

Note that the arrow marked with an exclamation mark is a weak equivalence because it is part of the Waldhausen weak equivalence square $(A \xrightarrow{\sim} A') \rightarrowtail (B \xrightarrow{\sim} B')$, and therefore (2) is a weak equivalence. Then $B \cup_A C \xrightarrow{\sim} B \cup_A C' = C' \cup_A B \xrightarrow{\sim} B' \cup_{A'} C'$ is the desired weak equivalence. ☐

Definition 4.10. Let $X_\bullet$ be a simplicial space. The space $L_n X = \bigcup_{i=0}^{n-1} s_i(X_{n-1}) \subseteq X_n$ is called the *n-th latching object of* $X_\bullet$, and the inclusion $L_n X \hookrightarrow X_n$ is called the *n-th latching map*. If $L_n X$ is a cofibration for all $n \in \mathbb{N}$, then $X_\bullet$ is called *Reedy-cofibrant.* ⌟

Proposition 4.11. *A simplicial CGWH-space $X_\bullet$ is Reedy-cofibrant if and only if all degeneracies s_i are cofibrations.*

Proof. We show by induction on n that $L_n X \rightarrowtail X_n$ is a cofibration iff all degeneracies are cofibrations. The case $n = 1$ is clear: $L_1 X = s_0(X_0) \rightarrowtail X_1$ iff s_0 is a cofibration. Assume $n > 1$, then we construct the latching object $L_n X$ using the following pushout (of inclusions) on the right repeatedly:

$$\begin{array}{ccccc}
\bigcup_{i=0}^{k-1} s_i(X_{n-2}) & \xrightarrow{s_k} & s_k\left(\bigcup_{i=0}^{k-1} s_i(X_{n-2})\right) & \hookrightarrow & \bigcup_{i=0}^{k-1} s_i(X_{n-1}) \\
\downarrow & & \downarrow & \ulcorner & \downarrow \\
X_{n-1} & \xrightarrow{s_k} & s_k(X_{n-1}) & \longrightarrow & \bigcup_{i=0}^{k} s_i(X_{n-1})
\end{array} \tag{5}$$

Indeed this is a pushout: Using the simplicial relations (1^{op}), for $x, x' \in X_{n-1}$ with $s_k(x) = s_i(x')$, $i < k$, we have $x = d_k s_k(x) = d_k s_i(x') = s_j(y)$ for some $y \in X_{n-2}$ and $j = i - 1$ or $j = i$.

Assume $X_\bullet$ is Reedy-cofibrant. Then for all k the map $\bigcup_{i=0}^{k-1} s_i(X_{n-1}) \rightarrowtail \bigcup_{i=0}^{k} s_i(X_{n-1})$ is a cofibration: In the base case $k = 0$, the diagram is the pushout of $s_0(X_{n-1}) \leftarrow \varnothing \to \varnothing$, and trivially $\varnothing \rightarrowtail s_0(X_{n-1})$ is a cofibration. By induction hypothesis on n and Reedy-cofibrancy, there is a composite cofibration $\bigcup_{i=0}^{k-1} s_i(X_{n-2}) \rightarrowtail L_{n-1}X \rightarrowtail X_{n-1}$. The s_i are homeomorphisms onto their image, and by cobase change we obtain $\bigcup_{i=0}^{k-1} s_i(X_{n-1}) \rightarrowtail \bigcup_{i=0}^{k} s_i(X_{n-1})$. Thus the composition $s_i : X_{n-1} \rightarrowtail L_nX \rightarrowtail X_n$ is a cofibration for all i.

Now assume all degeneracies are cofibrations. Again by induction on $k \leq n$ and cobase change we directly obtain cofibrations $\bigcup_{i=0}^{k-1} s_i(X_{n-1}) \rightarrowtail \bigcup_{i=0}^{k} s_i(X_{n-1})$. We inductively construct a cofibration $L_nX \rightarrowtail X_n$: By induction hypothesis on k, $\bigcup_{i=0}^{k-1} s_i(X_{n-1}) \rightarrowtail X_n$ is a cofibration, and by assumption $s_k(X_{n-1}) \rightarrowtail X_n$ is a cofibration. By the Waldhausen cofibration square property, $\bigcup_{i=0}^{k} s_i(X_{n-1}) \rightarrowtail X_n$ is a cofibration for all k. In particular, $L_nX \rightarrowtail X_n$ is a cofibration. □

Definition 4.12. Let $X_\bullet$ be a simplicial space. The *n-skeleton of the realization* is, for $n \in \mathbb{N}$, defined as

$$\mathrm{Sk}_n\,|X_\bullet| = \mathrm{coequ}\left(\coprod_{f\in \mathrm{Mor}\,\Delta|_n} X_{\mathrm{cod}\,f} \times |\Delta^{\mathrm{dom}\,f}| \underset{f^{\mathrm{op}}\times \mathrm{id}}{\overset{\mathrm{id}\times f}{\rightrightarrows}} \coprod_{m\in \mathrm{Ob}(\Delta|_n)} X_m \times |\Delta^m| \right),$$

where $\Delta|_n$ is the full subcategory of Δ with objects $[0],\dots,[n]$ (and hence $f \in \mathrm{Mor}\,\Delta|_n$ implies $\mathrm{dom}\,f, \mathrm{cod}\,f \subseteq [n]$). ⌟

Proposition 4.13. *The skeleta can be obtained iteratively via the following pushout:*

$$\begin{array}{ccc} P_n^X & \longrightarrow & \mathrm{Sk}_{n-1}\,|X_\bullet| \\ \downarrow & \ulcorner & \downarrow \\ X_n \times |\Delta^n| & \longrightarrow & \mathrm{Sk}_n\,|X_\bullet| \end{array}$$

Proof. It is enough to consider the non-degenerate simplices of dimension n. Note that we have coverings $\coprod_{s_i} X_{n-1} \times |\Delta^n| \to L_n \times |\Delta^n|$ and $\coprod_{d_i} X_n \times |\Delta^{n-1}| \to X_n \times \partial|\Delta^n|$. We define

$$P_n^X := L_nX \times |\Delta^n| \cup_{L_nX\times\partial|\Delta^n|} X_n \times \partial|\Delta^n| \subseteq X_n \times |\Delta^n|.$$

The map to $\mathrm{Sk}_{n-1}\,|X_\bullet|$ is determined by the canonical maps into the skeleton, which is a quotient:

$$\begin{aligned} L_nX \times |\Delta^n| \to \mathrm{Sk}_{n-1}\,|X_\bullet|, &\quad (s_ix, p) \mapsto (x, S_ip) \in X_{n-1} \times |\Delta^{n-1}| \\ X_n \times \partial|\Delta^n| \to \mathrm{Sk}_{n-1}\,|X_\bullet|, &\quad (x, D_ip) \mapsto (d_ix, p) \in X_{n-1} \times |\Delta^{n-1}| \end{aligned}$$

□

Because colimits commute, we immediately get:

Corollary 4.14. $|X_\bullet| \cong \operatorname*{colim}_n \mathrm{Sk}_n\,|X_\bullet|$.

Theorem 4.15. *Let $f_\bullet : X_\bullet \to Y_\bullet$ be a map of* good *simplicial CGWH-spaces, such that $f_n\colon X_n \to Y_n$ is a weak equivalence for all $n \in \mathbb{N}$. Then $|f_\bullet| : |X_\bullet| \to |Y_\bullet|$ is a weak equivalence.*

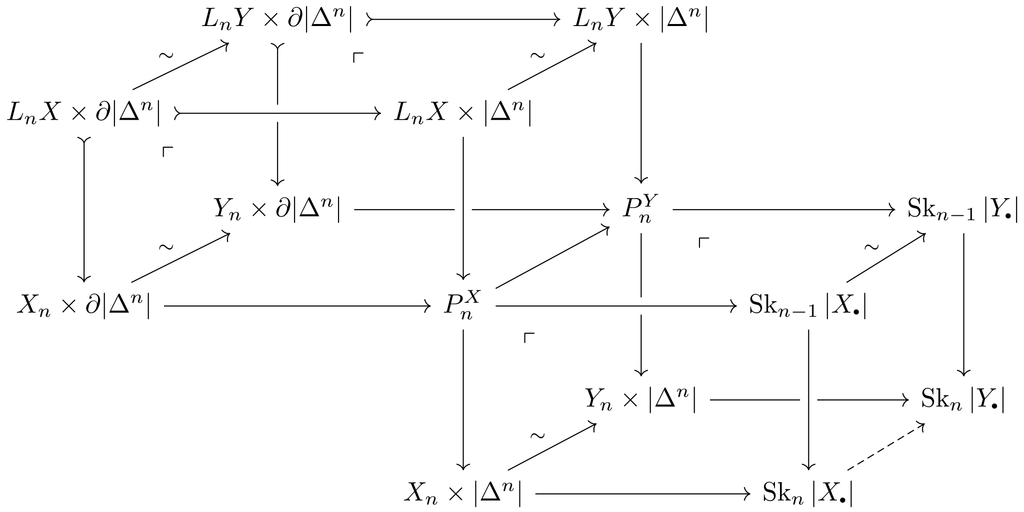
Proof. This proof follows the sketch given in [Dug08, Thm. 3.5].

Step 1: There exists a weak equivalence $L_nX \to L_nY$ for all n.
This is clear in for the cases $L_0X = \varnothing = L_0Y$ and $L_1X = s_0(X_0) \cong X_0 \xrightarrow{f_0} Y_0 \cong s_0(Y_0) = L_1Y$. For $n \geq 2$, we use the inductive construction diagram (5) from proposition 4.11 above for the latching objects L_nX and L_nY: There is an obvious weak equivalence $s_k(X_{n-1}) \to s_k(Y_{n-1})$ via f_{n-1} for all $k \leq n$ by assumption. Inductively, the gluing lemma yields (unique) weak equivalences $\bigcup_{i=0}^{k} s_i(X_{n-1}) \to \bigcup_{i=0}^{k} s_i(Y_{n-1})$ for all $k \leq n$. In the case $k = n$ we obtain the desired equivalence $L_nX \to L_nY$.

Step 2: There exists a weak equivalence of the n-skeleta of $|X_\bullet|$ and $|Y_\bullet|$.

Again, this is obvious in the case $n = 0$, where we have $\mathrm{Sk}_0\,|X_\bullet| \cong X_0 \times \{*\} \xrightarrow{f_0\times\mathrm{id}} Y_0 \times \{*\} \cong \mathrm{Sk}_0\,|Y_\bullet|$, and by assumption f_0 is a weak equivalence.

For the induction step, consider the following diagram. The front and back rectangles are pushouts, and the diagonal maps that are marked as weak equivalences are the obvious ones obtained either via the latching object equivalence from step 1, the given weak equivalence f_n, or by induction hypothesis on the $(n-1)$-skeleta.



By assumption all degeneracies are cofibrations, and proposition 4.11 implies that the latching maps are cofibrations; obviously, $\partial|\Delta^n| \rightarrowtail |\Delta^n|$ is a cofibration, too. Notice that the upper left pushout together with the two maps $L_nX \times |\Delta^n| \rightarrowtail X_n \times |\Delta^n|$ and $X_n \times \partial|\Delta^n| \rightarrowtail X_n \times |\Delta^n|$ is a Waldhausen cofibration square, and thus $P_n^X \rightarrowtail X_n \times |\Delta^n|$ is a cofibration; it follows that $\mathrm{Sk}_{n-1}\,|X_\bullet| \rightarrowtail \mathrm{Sk}_n\,|X_\bullet|$ is a cofibration, too. Exchanging X and Y yields mirror statements.

The gluing lemma for the upper left cube yields a weak equivalence $P_n^X \xrightarrow{\sim} P_n^Y$, and applying it again for the lower right cube, we obtain the dashed arrow as (unique) weak equivalence $\mathrm{Sk}_n\,|X_\bullet| \xrightarrow{\sim} \mathrm{Sk}_n\,|Y_\bullet|$.

Step 3: We have seen that the skeleta include as cofibrations, and in CGWH this implies (cf. [Str09, Lemma 3.6]) that the diagram

$$\mathrm{Sk}_0\,|X_\bullet| \rightarrowtail \mathrm{Sk}_1\,|X_\bullet| \rightarrowtail \cdots \rightarrowtail \mathrm{Sk}_n\,|X_\bullet| \rightarrowtail \mathrm{Sk}_{n+1}\,|X_\bullet| \rightarrowtail \cdots$$

is strongly filtered, i. e. any continuous $K \to \mathrm{colim}_n\,\mathrm{Sk}_n\,|X_\bullet|$, K compact, factors through $\mathrm{Sk}_i\,|X_\bullet|$ for some i. In particular this is true for the n-spheres S^n. Thus $|X_\bullet|$ and $|Y_\bullet|$ are weakly equivalent. □

An analogous result is true for levelwise cofibrations:

Theorem 4.16. *Let $f_\bullet : X_\bullet \to Y_\bullet$ be a map of* good *simplicial CGWH-spaces, such that $f_n\colon X_n \to Y_n$ is a cofibration for all $n \in \mathbb{N}$. Then $|f_\bullet| : |X_\bullet| \to |Y_\bullet|$ is a cofibration.*

Proof. The previous proof relies mainly on the gluing lemma for weak equivalences (4.9), for which we shall establish an analogous result in the case that the diagonal maps α, β and γ are cofibrations: Then δ is also a cofibration.

We need not prove this manually, but instead employ machinery developed by Waldhausen for [Wal85, Lemma 1.1.1]. The category $F_1\mathsf{CGWH}$ has as objects cofibrations $A \rightarrowtail B$ between CGWH-spaces, and morphisms are the obvious commutative squares. The cofibrations co F_1CGWH of this category are the Waldhausen cofibration squares, i. e. those squares $(A \rightarrowtail B) \to (A' \rightarrowtail B')$ with $A \rightarrowtail A'$ and $A' \cup_A B \rightarrowtail B'$ CGWH-cofibrations. The lemma states that co F_1CGWH makes F_1CGWH into a category with cofibrations, in particular the pushouts exist.

Take the same cube diagram as in the gluing lemma 4.9, but suppose α, β and γ are cofibrations and $A = A' \cap C$. By lemma 4.8 (ii) the square of cofibrations is a Waldhausen cofibration square, so the left vertical map is a cofibration in F_1CGWH. Thus the cube diagram translates to the following pushout diagram in F_1CGWH:

$$\begin{array}{ccc} (A \overset{\alpha}{\rightarrowtail} A') & \longrightarrow & (B \overset{\beta}{\rightarrowtail} B') \\ \downarrow & \ulcorner & \downarrow \\ (C \overset{\gamma}{\rightarrowtail} C') & \longrightarrow & (D \overset{\delta}{\rightarrowtail} D') \end{array}$$

Important is not that the stronger fact that the righthand vertical map is a cofibration; it is enough that the pushout *object* exists, since this already is the desired cofibration of CGWH-spaces $D \rightarrowtail D'$.

With the gluing lemma alone we obtain:

1. The latching maps $L_n X \rightarrowtail L_n Y$ are cofibrations and
2. The skeleta include as cofibrations $\mathrm{Sk}_n |X_\bullet| \rightarrowtail \mathrm{Sk}_n |Y_\bullet|$.

Lastly, we need to see that the cofibrations are stable under taking the (sequential) colimit. We have a "ladder of cofibrations"

$$\begin{array}{ccccccccc} \cdots & \rightarrowtail & \mathrm{Sk}_{n-1}|X_\bullet| & \rightarrowtail & \mathrm{Sk}_n|X_\bullet| & \rightarrowtail & \mathrm{Sk}_{n+1}|X_\bullet| & \rightarrowtail & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \rightarrowtail & \mathrm{Sk}_{n-1}|Y_\bullet| & \rightarrowtail & \mathrm{Sk}_n|Y_\bullet| & \rightarrowtail & \mathrm{Sk}_{n+1}|Y_\bullet| & \rightarrowtail & \cdots \end{array}$$

and lemma 4.17 below implies that $|X_\bullet| \rightarrowtail |Y_\bullet|$ is a cofibration, completing the proof. □

Lemma 4.17. *Let $A_0 \rightarrowtail A_1 \rightarrowtail A_2 \rightarrowtail \cdots$ and $B_0 \rightarrowtail B_1 \rightarrowtail B_2 \rightarrowtail \cdots$ be two sequential diagram of cofibrations, and denote their respective colimits by $A = \operatorname{colim} A_i$ and $B = \operatorname{colim} B_i$. Then:*

1. *For all $i \in \mathbb{N}$, $A_i \rightarrowtail A$ and $B_i \rightarrowtail B$ are cofibrations.*
2. *If $A_i \rightarrowtail B_i$ are cofibrations for all $i \in \mathbb{N}$, then the induced map $A \rightarrowtail B$ is a cofibration.*

Proof. By truncating the sequence, it is enough to prove $A_0 \rightarrowtail A$ is a cofibration. Assume a homotopy $H_0 : A_0 \times I \to X$ and a map $G : A \to X$ with $G|_{A_0} = H_0(-, 0)$. We want to extend H_0 to a homotopy $H : A \times I \to X$ such that $H(-, 0) = G$.

We can construct the extension $H_1 : A_1 \times I \to X$ with $H_1(-, 0) = G|_{A_1}$ by using the homotopy extension property for the cofibration $A_0 \rightarrowtail A_1$. Inductively, we obtain homotopies $H_n : A_n \times I \to X$ for all n. The space A carries the final topology, so the inclusions $a_i : A_i \to A$ are continuous, and a map $f : A \to X$ is continuous iff $f \circ a_i$ is continuous for all i. Define the map H as the union of all H_i, $i \geq 0$. This is well-defined, and since $H \circ (a_i \times \mathrm{id}_I) = H_i$ for all i, H is continuous. Clearly $H|_{A_0 \times I} = H_0$ and $H(-, 0) = G$, so H extends H_0 in the desired way. Thus the HEP is satisfied, and $A_0 \rightarrowtail A$ is a cofibration.

For the second statement, consider a homotopy $H_A : A \times I \to X$, which we wish to extend along some $G : B \to X$. For each i, we have a homotopy $H_{A,i} = H_A \circ (a_i \times \mathrm{id}_I) : A_i \times I \to X$ which we can extend via the cofibration $A_i \rightarrowtail B_i$ to a homotopy $H_{B,i}$ with $H_{B,i}(-, 0) = G|_{B_i}$. Again, the union of the $H_{B,i}$ constitutes the desired homotopy $H_B : B \times I \to X$ that extends H_A and has $H_B(-, 0) = G$. □

Corollary 4.18. *Given a levelwise cofibration sequence of good simplicial spaces $A_\bullet \rightarrowtail X_\bullet \twoheadrightarrow X_\bullet/A_\bullet$, the sequence $|A_\bullet| \rightarrowtail |X_\bullet| \twoheadrightarrow |X_\bullet/A_\bullet|$ is a cofibration sequence, too.*

Proof. By [Str09, 2.17], the equivalence relation $\sim$ is compatible with realization in the sense that the natural map $|X_\bullet/A_\bullet| \to |X_\bullet|/|A_\bullet|$ is a homeomorphism. □

5 Realizations of Cyclic Spaces exhibit an $\mathrm{SO}(2)$-action

So far we have only looked at the realization of simplicial spaces. This section establishes the important fact that realizations of cyclic spaces carry a canonical "circle action" of the group SO(2).

Definition 5.1. Let $X_\bullet$ be a cyclic space. The *realization of* $X_\bullet$ is defined via the composition

$$\mathsf{CycSpace} \xrightarrow{j^*} \mathsf{SimpSpace} \xrightarrow{|-|} \mathsf{CGWH},$$

i. e. as the realization of the underlying simplicial space. For short, one writes $|X_\bullet|$ for $|j^*X_\bullet|$. ⌟

Note. The proof of theorem 5.9 provides a rationale why one need not define a specific realization functor for cyclic spaces.

Remark 5.2. We introduce new notation for the common situation of switching between different canonical representations of morphisms in Λ.

Given a morphism $f \circ g \in \Lambda^n[m]$ with $f \in \mathrm{Hom}_\Delta([m],[n])$ and $g \in C_{m+1}^{\mathsf{op}}$, then by applying the cocyclic relations, there exists a unique decomposition

$$f \circ g = f^{\leftarrow}(g) \circ g^{\rightarrow}(f), \quad \text{with } f^{\leftarrow}(g) \in C_{n+1}^{\mathsf{op}} \text{ and } g^{\rightarrow}(f) \in \mathrm{Hom}_\Delta([m],[n]).$$

For example, in the case $f = \sigma_n$, $g = \tau_{n+1}^2$, we have $\sigma_n\tau_{n+1}^2 = \tau_n\sigma_0 = f^{\leftarrow}(g)g^{\rightarrow}(f)$. For the analogous case $g \circ f$ with $f \in \mathrm{Hom}_\Delta([m],[n])$ and $g \in C_{n+1}^{\mathsf{op}}$, we write:

$$g \circ f = g^{\rightarrow}(f) \circ f^{\leftarrow}(g), \quad \text{with } g^{\rightarrow}(f) \in \mathrm{Hom}_\Delta([m],[n]) \text{ and } f^{\leftarrow}(g) \in C_{m+1}^{\mathsf{op}}.$$

As a mnemonic device, think of $f^{\leftarrow}(g)$ as "g, after an f was moved from right of g to the left of g". ⌟

Example 5.3. As a motivating example and because it will play a crucial role in later proofs, we will study the *cyclic set of cyclic groups* $C_\bullet$ in detail (cf. [Lod98, 6.1.9–10]). It consists of the (opposite) cyclic groups $(C_{n+1}^{\mathsf{op}})_{n\in\mathbb{N}}$ (regarded as sets), and the morphisms induced by the unique decomposition presented in prop. 1.10: Let $f \circ g \in \mathrm{Hom}_\Delta([m],[n]) \times C_{m+1}^{\mathsf{op}} \cong \mathrm{Hom}_\Lambda([m],[n])$. Then applying the structure map $(f \circ g)^{\mathsf{op}} : [n] \to [m]$ to $c \in C_{n+1}^{\mathsf{op}}$ yields:

$$(f \circ g)^{\mathsf{op}}(c) = c \circ f \circ g = c^{\rightarrow}(f) \circ f^{\leftarrow}(c) \circ g,$$

where $f^{\leftarrow}(c) \circ g \in C_{m+1}^{\mathsf{op}}$ is uniquely determined. The mapping $(f \circ g)^{\mathsf{op}}(c) \mapsto f^{\leftarrow}(c) \circ g$ is functorial, and hence we have a cyclic structure on $C_\bullet$.

To describe this structure more explicitly, we say what the torsion, face and degeneracy maps do. Given $c \in C_{n+1}^{\mathsf{op}}$, t_n acts in the obvious way by multiplication, and we have $t_n^{n+1}(c) = c\tau_n^{n+1} = c$. Hence it's justified to think of τ_n as the generator of $C_{n+1}^{\mathsf{op}} \cong \mathrm{Aut}_{\Lambda^{\mathsf{op}}}([n]) = \langle\tau_n\rangle$.

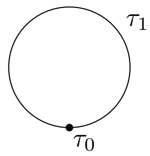
Using the cyclic relations (2^{op}), we can see what the face and degeneracy maps do on the generators τ_n:

$$d_i(\tau_n) = \tau_n\delta_i = \tau_{n-1}, \ \ 1 \le i \le n; \qquad d_0(\tau_n) = \mathrm{id}$$
$$s_i(\tau_n) = \tau_n\sigma_i = \tau_{n+1}, \ \ 1 \le i \le n; \qquad s_0(\tau_n) = t_{n+1}^2$$

Note in particular that the generators of $C_{n+1}^{\mathsf{op}} = \langle\tau_n\rangle$ for $n \ge 2$ are degenerate because $s_1(\tau_n) = \tau_{n+1}$, and thus all n-cells are degenerate for $n \ge 2$. ⌟

Note. We denote by $S^1 = \{z = a + ib \in \mathbb{C} \,||z| = 1\} \subset \mathbb{C}$ the (pointed) topological space, obtained for example as a CW-complex with one 0-cell and a single 1-cell. In contrast, to make clear that we are interested in the rotational group structure of the object, we use the (topological) group SO(2), also called the *circle group*. Since we will need to compare different circle actions, we will often write the circle component as $\mathbb{R}/a\mathbb{Z}$ for some a and silently assume the standard identification with SO(2).

Proposition 5.4. $|C_\bullet| = \mathbb{R}/\mathbb{Z} \cong S^1 \cong \mathrm{SO}(2)$



Proof of Proposition. In dimension 0, we have a single cell $\tau_0 = \mathrm{id}$. We have a 1-cell τ_1 which has $d_0(\tau_1) = \tau_0 = d_1(\tau_1)$, and since $s_0(\tau_0) = \tau_1^2 = \mathrm{id} \neq \tau_1$, this 1-cell is nondegenerate. The generators of all n-cells, $n \geq 2$, are degenerate, and therefore do not influence the realization. □

Remark 5.5. Since $\mathrm{Hom}_{\Lambda^{\mathsf{op}}}([m],[n]) \cong \mathrm{Hom}_{\Delta^{\mathsf{op}}}([m],[n]) \times C_{n+1}$, and $\Delta^0 = \mathrm{Hom}_{\Delta^{\mathsf{op}}}([0], -)$ is a singleton, we have $C_\bullet \cong \Lambda^0$. The realization of Δ^0 is a single point $\{*\}$, so the proposition shows that $|\Lambda^0| \cong \mathrm{SO}(2) \cong \mathrm{SO}(2) \times |\Delta^0|$. The realization $|\Lambda^0|$ admits an obvious SO(2)-action: multiplication on the left on SO(2), and trivial action on the other component; in fact this is true in all dimensions, as lemma 5.7 will show. The goal of this section is to generalize this to the realization of arbitrary cyclic spaces: They all carry a canonical SO(2)-action. ⌟

Remark 5.6. Since τ_1 generates $C_\bullet$, τ_m^i is the image of an application of degeneracies; in particular, in $C_\bullet$ we have

$$s_0^j(\tau_m^i) = \tau_m^i \sigma_0^j = \tau_m^{i-1}\sigma_m \tau_{m+1}^2 \sigma_0^{j-1} = \tau_{m+1}^{i-1}\tau_{m+1}^2\sigma_0^{j-1} = \tau_{m+1}^{i+1}\sigma_0^{j-1} = \cdots = \tau_{m+j}^{i+j}$$

and with $s_1^j(\tau_1) = \tau_1\sigma_1^j = \tau_{j+1}$ we obtain

$$s_0^{i-1}s_1^{m-i}(\tau_1) = s_0^{i-1}(\tau_{m-i+1}) = \tau_m^i.$$

The element $\tau_m^0 = \tau_m^{m+1} = \mathrm{id}$ is generated by $s_1^m s_0 d_0(\tau_1)$.

Thus in the quotient $|\Lambda^0| = \coprod C_{m+1} \times |\Delta^m| / \sim$, a point $[\tau_m^i, (p_0, \ldots, p_m)]$ has a canonical representative using the generator τ_1 of $C_\bullet$, namely

$$\begin{aligned}
i = 0, m+1: \qquad & [s_1^m s_0 d_0(\tau_1), (p_0, \ldots, p_m)] = [\tau_1, (0, p_0 + \cdots + p_m)] = [\tau_1, (0, 1)] \\
1 \leq i \leq m: \qquad & [s_0^{i-1}s_1^{m-1}\tau_1, (p_0, \ldots, p_m)] = [\tau_1, S_1^{m-1}S_0^{i-1}(p_0, \ldots, p_m)] \\
& \qquad = [\tau_1, (p_0 + \cdots + p_{i-1}, p_i + \cdots + p_m)]
\end{aligned}$$

or equivalently, by identifying $|\Lambda^0| = |C_\bullet| \cong \mathbb{R}/\mathbb{Z}$ through projection onto the second coordinate, $p_i + \cdots + p_m + \mathbb{Z} = 1 - (p_0 + \cdots + p_{i-1}) + \mathbb{Z} =: \theta + \mathbb{Z} \in \mathbb{R}/\mathbb{Z}$. ⌟

Lemma 5.7. *There exists an isomorphism of cocyclic spaces* $|\Lambda^\bullet| \cong \mathrm{SO}(2) \times |\Delta^\bullet|$. *The cocyclic structure maps are equivariant with respect to the canonical* SO(2)*-action.*

Proof. We first establish the homeomorphism $|\Lambda^n| \cong \mathrm{SO}(2) \times |\Delta^n|$ for all n and then proceed to show that the objects are compatible with the cocyclic structure maps.

Consider the map $h_n : |\Lambda^n| \to |\Lambda^0 \times \Delta^n|$ given by sending $[x \circ g, p] \mapsto [(g^{-1}, x), g(p)]$, where when applying an element g (or, in the proof, $f^{\leftarrow}(g) \in C_{m+1}^{\mathsf{op}}$), we use the cocyclic structure of "rotation of corners" on $|\Delta^m|$ (cf. example 1.9). The map is well-defined: Given $f \in \mathrm{Hom}_\Delta([m],[m'])$, we have[4]

$$\begin{aligned}
h_n[f(x \circ g); p] &= h_n[x \circ g \circ f; p] \\
&= h_n[x \circ g^{\rightarrow}(f) \circ f^{\leftarrow}(g); p]
\end{aligned}$$

[4] Note that we use a shorthand notation here, e. g. a statement $[f(x), p] = [x, f(p)]$ would read, if spelled out explicitly, $[\Delta^n[f^{\mathsf{op}}](x), p] = [x, |\Delta^n[f]|(p)]$.

$$
\begin{aligned}
&= [f^{\leftarrow}(g)^{-1}, x \circ g^{\rightarrow}(f); f^{\leftarrow}(g)(p)] \\
&\overset{(*)}{=} [g^{-1} \circ g^{\rightarrow}(f), x \circ g^{\rightarrow}(f); f^{\leftarrow}(g)(p)] \\
&= [g^{-1}, x; g^{\rightarrow}(f) f^{\leftarrow}(g)(p)] \\
&= [g^{-1}, x; gf(p)] \\
&= h_n[x \circ g; f(p)].
\end{aligned}
$$

The equality marked $(*)$ holds because of $gf = g^{\rightarrow}(f)f^{\leftarrow}(g) \Leftrightarrow ff^{\leftarrow}(g)^{-1} = g^{-1}g^{\rightarrow}(f)$, and $ff^{\leftarrow}(g)^{-1}$ in $C_\bullet = \Lambda^0$ equals $f^{\leftarrow}(g)^{-1}$. Clearly, the map h_n is a homeomorphism with inverse $h_n^{-1} : [(h, y), q] \mapsto [y \circ h^{-1}, h(q)]$.

The desired homeomorphism is given by the composition

$$
|\Lambda^n| \xrightarrow{h_n} |\Lambda^0 \times \Delta^n| \xrightarrow{\cong} |\Lambda^0| \times |\Delta^n| \xrightarrow{h_0 \times \mathrm{id}} |\Lambda^0| \times |\Delta^n|,
$$

which sends

$$
[x \circ \tau_m^i, (p_0, \dots, p_m)] \mapsto ([\tau_m^i, (p_0, \dots, p_m)], [x, \tau_m^i(p_0, \dots, p_m)])
$$

corresponding to the following point in $\mathrm{SO}(2) \times |\Delta^n|$:

$$
([\tau_1, (p_0 + \dots + p_{i-1}, p_i + \dots + p_m)], [x, (p_i, \dots, p_m, p_0, \dots, p_{i-1})]) =: (\theta, u),
$$

where $u = (u_0, \dots, u_n)$ is given by $[\mathrm{id}, x(p_i, \dots, p_m, p_0, \dots, p_{i-1})]$, i. e. relative to the generator of Δ^n. Applying t_n to $\Lambda^n[n]$ induces the map $T_n : (\theta; u_0, \dots, u_n) \mapsto (\theta - u_0; u_1, \dots, u_n, u_0)$, which clearly has $T_n^{n+1} = \mathrm{id}$.

The simplicial structure maps induced from d_i and s_i are the identity on the SO(2) component, and the usual maps on $|\Delta^n|$. One can easily see that the cocyclic relations are fulfilled, e. g.

$$
\begin{aligned}
T_n D_0(\theta; u_0, \dots, u_{n-1}) &= T_n(\theta; 0, u_0, \dots, u_{n-1}) = (\theta; u_0, \dots, u_{n-1}, 0) \\
&= D_n(\theta; u_0, \dots, u_{n-1}) \\
T_n S_0(\theta; u_0, \dots, u_{n+1}) &= T_n(\theta; u_0 + u_1, u_2, \dots, u_{n+1}) \\
&= (\theta - (u_0 + u_1); u_2, \dots, u_{n+1}, u_0 + u_1) \\
&= S_n(\theta - u_0 - u_1; u_2, \dots, u_{n+1}, u_0, u_1) \\
&= S_n T_{n+1}^2(\theta; u_0, u_1, \dots, u_{n+1}).
\end{aligned}
$$

The SO(2)-action on $\mathrm{SO}(2) \times |\Delta^n|$ is left-multiplication on the first, and the identity on the second coordinate: In particular T_n is SO(2)-equivariant. Thus the $\{\mathrm{SO}(2) \times |\Delta^n|\}_{n \in \mathbb{N}}$ assemble to a cocylic space $\mathrm{SO}(2) \times |\Delta^\bullet|$ with SO(2)-equivariant structure maps. □

Remark 5.8. The homeomorphism $|\Lambda^n| \cong \mathbb{R}/\mathbb{Z} \times |\Delta^n|$ is not as straight-forward as one might imagine at first. There are two reasons why we cannot go for a "simpler" approach.

First, given an element $f \circ g \in \Lambda^n[m]$, the projection $\Lambda^n[m] \to \Lambda^0[m], f \circ g \mapsto g$ is simplicial, but $\Lambda^n[m] \to \Delta^n[m], f \circ g \mapsto f$ is *not* (and hence the realization of this projection is not well-defined): for example, $d_0(\mathrm{id} \circ \tau_n) = \tau_n \delta_0 = \delta_n$ which is not the same as first projecting to id and then applying d_0, which yields δ_0.

Second, while the simplicial projections of the simplicial cartesian product induce a homeomorphism in the realization (cf. section 3), the simplicial set Λ^n cannot be obtained as a simplicial cartesian product, as it doesn't allow a cyclic structure: see figure 2. Thus we cannot pick an isomorphism of simplicial sets and take its realization. Some authors introduce a "twisted simplicial product" which induces a homeomorphism in the realization, see [DHK85]. ⌟

Theorem 5.9. *Let $X_\bullet$ be a cyclic space. Then $|X_\bullet|$ admits a canonical* SO(2)*-action.*

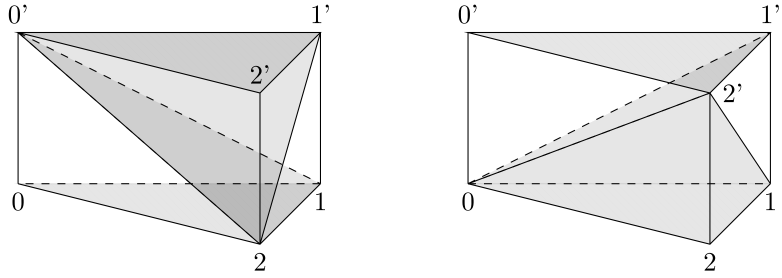


Figure 2: Two different triangulations of $\mathbb{R}/\mathbb{Z}\times|\Delta^2|$ (the dashed simplices are identified). The left one is Λ^2, and the right one represents $\Delta^1\times\Delta^2$ modulo the relation $[0,1,2]\equiv[0',1',2']$. As different triangulations their realization is homeomorphic, but the left one admits a cyclic structure while the simplicial set one the right does not allow for a cyclic structure that permutes the shaded simplices. On the left we have, in accordance with (2^{op}), $d_1t_2[0',1',2'] = d_1[2,0',1'] = [2,1'] = t_1[1',2'] = t_1d_0[0',1',2']$. On the right this fails, since $d_1t_2[0',1',2'] = d_1[0,1',2'] = [0,2']$, but $t_1d_0[0',1',2'] = t_1[1',2'] = [1,2']$.

Proof. We have shown this in lemma 5.7 for the standard cocyclic space $|\Lambda^\bullet|$. Analogously to proposition 2.7, a cyclic space is "built up" from the prototypical standard cyclic sets, i. e. $X_\bullet \cong X_\bullet \times_\Lambda \Lambda^\bullet = \int^n X_n \times \Lambda^n$. Coends commute and the product preserves colimits, so we obtain:

$$\begin{aligned}|X_\bullet| &= \int^{m:\Delta} X_m \times |\Delta^m| \cong \int^{m:\Delta}\int^{n:\Lambda} X_n \times \Lambda^n[m]\times|\Delta^m| \\ &\cong \int^{n:\Lambda} X_n \times \int^{m:\Delta} \Lambda^n[m]\times|\Delta^m| \cong \int^{n:\Lambda} X_n \times |\Lambda^n| \\ &\cong \int^{n:\Lambda} X_n \times \mathrm{SO}(2)\times|\Delta^n|.\end{aligned}$$

Put differently, the realization is obtained as the quotient

$$|X_\bullet| = \coprod_{n=0}^{\infty} X_n \times \mathrm{SO}(2)\times|\Delta^n| \Big/ \approx$$

where $\approx$ is the usual relation $\sim$ plus $(t_n x, p) \approx (x, T_n p)$, with T_n from the previous lemma. In terms of proposition 2.6, we look at the coequalizer with the sum indexed by $f \in \operatorname{Mor}\Lambda$, not only morphisms in Δ. Since T_n is SO(2)-equivariant, the action descends on the quotient. ☐

6 Edgewise Subdivision

The homotopy theory of G-spaces with G-equivariant maps is defined via ordinary homotopy theory on the fixed sets under $H \subseteq G$ for all (closed) subgroups H (see e. g. [Wan80]). In particular, a map $f : X \to Y$ is a G-weak equivalence if and only if it induces isomorphisms $\pi_n(X^H) \cong \pi_n(Y^H)$ for all subgroups H. Thus in the case $G = \mathrm{SO}(2)$ we are interested in the realization's fixed points under actions induced by subgroups of SO(2), and there are two cases: SO(2) itself, and the (discrete) subgroups isomorphic to $\mathbb{Z}/n\mathbb{Z}$.

Proposition 6.1. *Let $X_\bullet$ be a cyclic space. The fixed points of its realization under the canonical* SO(2)*-action are* $|X_\bullet|^{\mathrm{SO}(2)} = \{x \in X_0 \mid s_0x = t_1s_0x\}$.

Proof. The SO(2)-action on the $\mathrm{SO}(2)\times|\Delta^n|$ factors/summands of $|X_\bullet|$ is nontrivial; hence the only way the action can become trivial in the quotient is if the SO(2) part is being collapsed to a point. Since T_n is the only non-identity relation on SO(2), and all degenerate $|\Delta^n|$ that collapse to a point in the quotient are identified with a 0-cell $x \in X_0$, it is enough to consider T_1 which sends $(z; x_0, x_1) \mapsto (z - x_0; x_1, x_0)$.

Let $[x; z; 1]$ be a point in $\coprod X_n \times \mathrm{SO}(2) \times |\Delta^n|/\approx$. For all $a \in \mathrm{SO}(2)$, we must have $[x; z; 1] = [x; z - a; 1]$. We can rewrite

$$\begin{aligned} [x; z; 1] &= [x; S_0(z; a, 1-a)] \quad \text{and} \\ [x; za; 1] &= [x; S_0(z-a; 1-a, a)] = [x; S_0(T_1(z; a, 1-a))]. \end{aligned}$$

Using the relation $\approx$, we "switch around" the degeneracies and obtain

$$[s_0 x; z; a, 1-a] = [t_1 s_0 x; z; a, 1-a].$$

We see that for degenerate 1-cells $y = s_0 x$ with $t_1 y = y$ the factor SO(2) and $|\Delta^1|$ collapse to a point; Thus it makes sense to consider $\{y \in s_0(X_0) \mid t_1 y = y\} \subset |X_\bullet|$, and these are exactly the SO(2)-invariant points. □

Example 6.2. We saw in example 5.3 that $|C_\bullet| \cong \mathrm{SO}(2)$ with SO(2) acting from the left. Thus we should expect no fixed points under this action, and the proof of proposition 5.4 showed that $C_0 = \{t_0\}$, $s_0(t_0) = \mathrm{id}$, and $\mathrm{id} \neq t_1 \in C_1$, so indeed $|C_\bullet|^{\mathrm{SO}(2)} = \varnothing$ as expected. ⌟

In order to find the fixed sets of actions of the discrete subgroups $\mathbb{Z}/n\mathbb{Z}$, we need an elaborate subdivision.

Definition 6.3. For $a \geq 1$, define the *a-subdivision functor* $\mathrm{sd}_a : \Delta \to \Delta$ via

$$[n] \mapsto \coprod_{i=1}^{a} [n] = [a(n+1)-1], \qquad (f : [m] \to [n]) \mapsto (\coprod_{i=1}^{a} f : \coprod_{i=1}^{a} [m] \to \coprod_{i=1}^{a} [n]),$$

or equivalently $\mathrm{sd}_a(f) : [a(m+1)-1] \to [a(n+1)-1]$ by mapping $f : i(m+1)+r \mapsto i(n+1)+f(r)$ for $1 \leq i \leq a-1$ and $0 \leq r \leq m$. Given a simplicial space $X_\bullet$, we define $\mathrm{sd}_a X_\bullet$ as the composition $X_\bullet \circ \mathrm{sd}_a$. This procedure is called *edgewise subdivision.* ⌟

Remark 6.4. It is clear from the definition that the r-simplices of the a-subdivided $X_\bullet$ are exactly the $(a(r+1)-1)$-simplices of $X_\bullet$, in formulas $(\mathrm{sd}_a X_\bullet)_r = X_{a(r+1)-1}$. The face and degeneracy maps are as follows:

$$\begin{aligned} \mathrm{sd}_a X(d_i) &= d_i \circ d_{(n+1)+i} \circ d_{2(n+1)+i} \circ \cdots \circ d_{(a-1)(n+1)+i} : (\mathrm{sd}_a X_\bullet)_n \to (\mathrm{sd}_a X_\bullet)_{n-1} \\ \mathrm{sd}_a X(s_i) &= s_{(a-1)(n+1)+i} \circ \cdots \circ s_{2(n+1)+i} \circ s_{(n+1)+i} \circ s_i : (\mathrm{sd}_a X_\bullet)_n \to (\mathrm{sd}_a X_\bullet)_{n+1} \end{aligned}$$

⌟

Example 6.5. We can explicitly compute and draw 2- or 3-subdivisions of low-dimensional and easy-to-understand simplicial sets. Figure 3 shows two nontrivial subdivisions of Δ^2. For the 2-subdivision, vertices, edges and faces are order-preserving maps $[1] \to [2]$, $[3] \to [2]$, and $[5] \to [2]$, respectively. Only non-degenerate simplices are drawn; for example, the 1-simplex $[0, 0, 2, 2] = s_0[0, 2]$ is degenerate (and hence doesn't appear), whereas $[0, 0, 0, 2]$ can not be obtained from any vertex via the only 0-degeneracy s_0; it sits between the vertices $d_1[0, 0, 0, 2] = [0, 0]$ and $d_0[0, 0, 0, 2] = [0, 2]$.

We can still draw reasonable pictures in one dimension higher: The 2-subdivision of Δ^3 has vertices $(\mathrm{sd}_2 \Delta^3)_0 = \Delta^3[1]$, edges $(\mathrm{sd}_2 \Delta^3)_1 = \Delta^3[3]$, faces $(\mathrm{sd}_2 \Delta^3)_2 = \Delta^3[5]$ and 3-simplices $(\mathrm{sd}_2 \Delta^3)_3 = \Delta^3[7]$; all higher-dimensional simplices are degenerate. Figure 4 features a drawing of (the non-degenerate simplices of) $\mathrm{sd}_2(\Delta^3)$; it is suggestively drawn so that it should be obvious that $|\mathrm{sd}_2(\Delta^3)| \cong |\Delta^3|$. This is in fact true for any a-subdivision of any Δ^k, as will be shown in the next lemma. ⌟

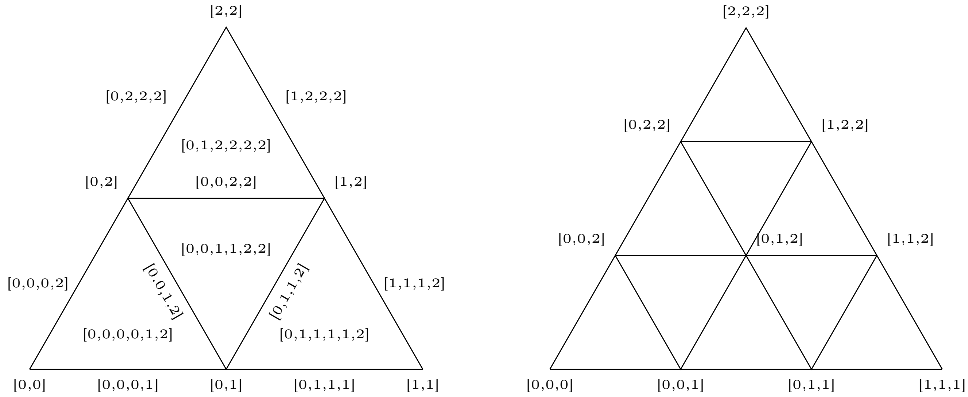


Figure 3: The subdivisions $\mathrm{sd}_2\,\Delta^2$ and $\mathrm{sd}_3\,\Delta^2$ in comparison. For simplicity, in the latter case only the vertices are prescribed: A 2-simplex is an order-preserving map $[8] \to [2]$; e. g. the top triangle is the map with image $[0,1,2,2,2,2,2,2,2]$.

Lemma 6.6. *Let $X_\bullet$ be a simplicial space and $a \geq 1$. Then there exists a homeomorphism $D_a : |\,\mathrm{sd}_a\,X_\bullet| \xrightarrow{\cong} |X_\bullet|$.*

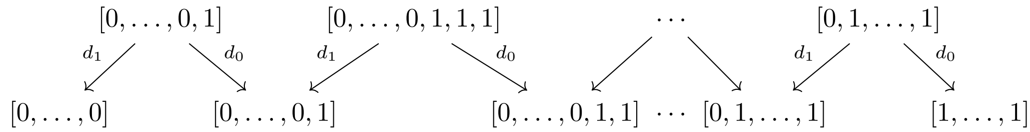
Note. If the context is clear, this homeomorphism will simply be denoted D. In cases where this notation is ambiguous, the notation D_a makes clear which subdivision is being "undone".

Proof. This proof is explicitly works out the steps indicated in the four-line "proof" of [BHM93, Lemma 1.1].

We will let D be the homomorphism induced by $\mathrm{id} \times d_a : X_{a(n+1)-1} \times |\Delta^n| \to X_{a(n+1)-1} \times |\Delta^{a(n+1)-1}|$ with $d_a(p) = \frac{1}{a}(p,\dots,p)$. For the coface and codegeneracies of the standard simplex we have $d_a \circ D_i(p) = \mathrm{sd}_a(D_i) \circ d_a(p)$ and $d_a \circ S_i(p) = \mathrm{sd}_a(S_i) \circ d_a(p)$, hence the map $\mathrm{id} \times d_a$ is well-defined and descends on the quotient.

Step 1: The claim holds true for $X_\bullet = \Delta^1$.

We can compute $\mathrm{sd}_a\,\Delta^1$ explicitly: It has $(a+1)$ vertices, namely all order-preserving maps $\mathrm{Hom}_\Delta([a-1],[1])$; we can represent a 0-cell as an a-tuple $[0,\dots,0,1,\dots,1]$. The 1-simplices of $\mathrm{sd}_a\,\Delta^1$ are all order-preserving maps $\mathrm{Hom}_\Delta([2a-1],[1])$ and we can likewise represent them as $2a$-tuples. The degeneracy sends $s_0 : [0,\dots,0,1,\dots,1] \mapsto [0,0,\dots,0,0,1,1,\dots,1,1]$, hence a 1-simplex is degenerate iff it has an even amount of zeros. The realization collapses degenerate simplices, so it is enough to consider the non-degenerate 1-cells. It is clear that all k-simplicies for $k \geq 2$ are degenerate. Thus we can picture the non-degenerate simplices in $\mathrm{sd}_a\,\Delta^1$ as follows:



The realization $|\Delta^1|$ has only one non-degenerate simplex in dimension ≥ 1, namely $\mathrm{id} \in \mathrm{Hom}_\Delta([1],[1])$. The 0-cell $v_i = [0_{(0)},\dots,0_{(i-1)},1_{(i)},\dots,1_{(a-1)}]$, $1 \leq i \leq a-1$, can be obtained as

$$v_i = \underbrace{s_0 \circ s_0 \circ \cdots \circ s_0}_{(i-1)\text{ times}} \circ \underbrace{s_1 \circ s_1 \circ \cdots \circ s_1}_{(a-i-1)\text{ times}} \circ\, \mathrm{id},$$

hence we have

$$D([v_i,1]) = [s_0^{i-1} s_1^{a-i-1}\,\mathrm{id}, d_a(1)] = [\mathrm{id}, S_1^{a-i-1} S_0^{i-1} d_a(1)] = [\mathrm{id}, (\frac{i}{a}, \frac{a-i}{a})],$$

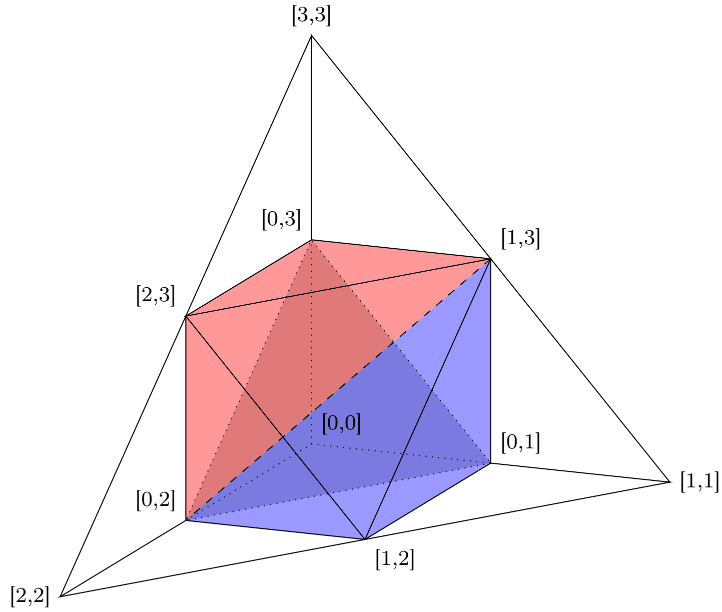


Figure 4: A drawing of $\mathrm{sd}_2\,\Delta^3$. To distinguish the simplices better, three of the overlapping ones are shaded in gray, red and blue; the two 3-simplices [0, 0, 0, 1, 1, 2, 3, 3] (with corners [0, 1], [0, 2], [0, 3], [1, 3]) and [0, 1, 1, 2, 2, 2, 3, 3] (with corners [0, 2], [1, 2], [1, 3], [2, 3]) are not shaded.

and regarding $|\Delta^1| \cong [0;1] \subseteq \mathbb{R}$ we can write $D(v_i) = \frac{i}{a}$. Thus we see that D "shrinks" the a one-simplices that are glued end-to-end in $\mathrm{sd}_a\,\Delta^1$ to intervals of length $\frac{1}{a}$ in $|\Delta^1| = [0;1]$.

Step 2: The claim holds true for $X_\bullet = \Delta^k$, $k \geq 2$.

We define maps $i : \Delta^k \hookrightarrow (\Delta^1)^{\times k}$ and $r : (\Delta^1)^{\times k} \twoheadrightarrow \Delta^k$ such that $r \circ i = \mathrm{id}$, i. e. r is a retract (one can think of this as decomposing a map into layers and then summing them up again):

$$i(\mathrm{id}_{[k]}) = (l_1, \ldots, l_k), \qquad l_j(m) = \begin{cases} 0 & \text{if} \quad 0 \leq m < k-j \\ 1 & \text{if} \quad k-j \leq m \leq k \end{cases}$$

$$r_m(f_1, \ldots, f_k) = \sum_{j=1}^{k} f_j, \qquad f_i : [m] \to [1]$$

The map $i = i_k$ is defined on the generator of Δ^k, and extends to a simplicial map by letting $i_m(f : [m] \to [k]) = (l_1 \circ f, \ldots, l_k \circ f)$. As an example, a (degenerate) 4-simplex in $\Delta^3[4]$ is mapped as follows:

$$[0,1,1,2,3] \xmapsto{i_4} ([0,1,1,1,1],[0,0,0,1,1],[0,0,0,0,1]) \xmapsto{r_4} [0,1,1,2,3].$$

The map r is simplicial, because we have

$$d_i(r_m(f_1, \ldots, f_k)) = \sum f_j \circ \delta_i = \sum d_i f_j = r_{m-1}(d_i(f_1, \ldots, f_k))$$

and similarly for the degeneracies s_i. In order to show that r is a retract, it is enough to show that $r \circ i$ is the identity on the generator $\mathrm{id}_{[k]} \in \Delta^k$, but this is obvious since $r(i(\mathrm{id}_{[k]}))(j) = j \cdot 1 = j$.

By functoriality we have retracts $|\,\mathrm{id}\,| = |r| \circ |i|$ and $|\,\mathrm{sd}_a(\mathrm{id})| = |\,\mathrm{sd}_a(r)| \circ |\,\mathrm{sd}_a(i)|$. Realization preserves products (see theorem 3.4), so we have the canonical homeomorphism $|(\Delta^1)^{\times k}| \cong |\Delta^1|^{\times k}$.

Consider the following diagram:

$$
\begin{array}{ccccc}
 & & |\operatorname{sd}_a((\Delta^1)^{\times k})| & & \\
 & \overset{|\operatorname{sd}_a(i)|}{\nearrow} & \downarrow \cong & \overset{|\operatorname{sd}_a(r)|}{\searrow} & \\
|\operatorname{sd}_a \Delta^k| & & |\operatorname{sd}_a(\Delta^1)^{\times k}| & & |\operatorname{sd}_a \Delta^k| \\
 & & \downarrow \cong & & \\
\downarrow D & & |\operatorname{sd}_a(\Delta^1)|^{\times k} & & \downarrow D \\
 & & D^{\times k} \downarrow \cong & & \\
|\Delta^k| & & |\Delta^1|^{\times k} & & |\Delta^k| \\
 & \underset{|i|}{\searrow} & \downarrow \cong & \underset{|r|}{\nearrow} & \\
 & & |(\Delta^1)^{\times k}| & &
\end{array}
$$

The left (and analogously, the right) quadrilaterals are commutative: Going down and right, we obtain for $f : [m] \to [k] \in \Delta^k[m]$:

$$[\textstyle\coprod_a f, p] \xmapsto{D} [\coprod_a f, d_a p] \xmapsto{|i|} [\prod_{i=1}^k \coprod_a f l_i, d_a p],$$

and going the other way we map

$$[\textstyle\coprod_a f, p] \xmapsto{|\operatorname{sd}_a(i)|} [\prod_{i=1}^k \coprod_a f l_i, p] \longmapsto [\coprod_a \prod_{i=1}^k f l_i, p] = [\prod_{i=1}^k \coprod_a f l_i, p] \longmapsto$$

$$\longmapsto \textstyle\prod_{i=1}^k [\coprod_a f l_i, p] \xmapsto{D^{\times k}} \prod_{i=1}^k [\coprod_a f l_i, d_a p] \longmapsto [\prod_{i=1}^k \coprod_a f l_i, d_a p].$$

The left quadrilateral shows that D is injective; the right one shows D is surjective. Thus D is a continuous bijection, and because $|\Delta^k|$ (and the relevant derived constructions like $|\operatorname{sd}_a \Delta^k|$) are compact CGWH-spaces, this implies D is a homeomorphism.

Step 3: The homeomorphism D is natural in $X_\bullet$, and we have

$$\operatorname{sd}_a(X_\bullet) = \operatorname{sd}_a\left(\int^n X_n \times \Delta^n\right) \cong \int^n X_n \times \operatorname{sd}_a \Delta^n.$$

Hence we obtain for an arbitrary simplicial space $X_\bullet$:

$$|\operatorname{sd}_a X_\bullet| = \int^n (\operatorname{sd}_a X_\bullet)_n \times |\Delta^n| \cong \int^n \int^m X_m \times \operatorname{sd}_a(\Delta^m) \times |\Delta^n| \cong \int^m X_m \times |\Delta^m| = |X_\bullet|$$

□

We will now go over to cyclic spaces, extend the sd_a functor and show that D is SO(2)-equivariant. First, we need an auxiliary category:

Definition 6.7. The category Λ_r, $r \geq 1$, is the category Δ with an additional generating arrow $\tau_n = \tau_{r,n}$ for all $n \in \mathbb{N}$ subject to the relations described in (2), except that the third line is replaced by $\tau_{r,n}^{r(n+1)} = \mathrm{id}$. ⌟

Remark 6.8. In particular, in the case $r = 1$ we have the original cyclic category Λ. Analogously to proposition 1.10, we see that $\operatorname{Hom}_{\Lambda_r}([m],[n]) \cong \operatorname{Hom}_\Delta([m],[n]) \times C^{\mathsf{op}}_{r(m+1)}$, and in particular $\operatorname{Aut}_{\Lambda_r}([n]) \cong C^{\mathsf{op}}_{r(n+1)} \cong \operatorname{Aut}_\Lambda([r(n+1)-1])$.

We can extend the edgewise subdivision functor $\mathrm{sd}_a : \Delta \to \Delta$ to a functor $\mathrm{sd}_a : \Lambda_{ra} \to \Lambda_r$, by letting it be the identity on $\mathrm{Aut}_{\Lambda_{ra}}([n]) \cong C^{\mathsf{op}}_{ra(n+1)} \to C^{\mathsf{op}}_{ra(n+1)} \cong \mathrm{Aut}_{\Lambda_r}([a(n+1)-1]) = \mathrm{Aut}_{\Lambda_r}(\mathrm{sd}_a[n])$. Writing j_r for the inclusion functor $\Delta \to \Lambda_r$, this can be pictured as:

$$\begin{array}{ccc} \Delta & \xrightarrow{\mathrm{sd}_a} & \Delta \\ \big\downarrow{\scriptstyle j_{ar}} & & \big\downarrow{\scriptstyle j_r} \\ \Lambda_{ar} & \xrightarrow{\mathrm{sd}_a} & \Lambda_r \end{array}$$

Again we have the *standard n-dimensional r-cyclic sets* Λ^n_r. ⌟

Example 6.9. We will compute the standard 2-cyclic set $\Lambda^1_2[-]$, see figure 5 for an illustration[5]. Since $\Lambda^1_2[-]$ has a single generator $\mathrm{id} : [1] \to [1]$, we can express its elements (uniquely) by applying face, degeneracy and torsion maps, which translate to precomposing with coface maps δ_i, codegeneracy maps σ_i and cotorsion maps $\tau_{2,n}$. We list the (not necessarily nondegenerate) cells in low dimensions by explicitly computing for $m = 0, 1, 2$ the expression $\Lambda^1_2[m] = \Delta^1[m] \times C^{\mathsf{op}}_{2(m+1)}$:

$$\begin{aligned} \Lambda^1_2[0] &= \{\delta_0, \delta_1\} \times \{\mathrm{id}, \tau_{2,0}\} \\ \Lambda^1_2[1] &= \{\delta_0\sigma_0, \mathrm{id}_{[1]}, \delta_1\sigma_0\} \times \{\mathrm{id}, \tau_{2,1}, \tau^2_{2,1}, \tau^3_{2,1}\} \\ \Lambda^1_2[2] &= \{\delta_0\sigma_0\sigma_0, \sigma_0, \sigma_1, \delta_1\sigma_0\sigma_0\} \times \{\mathrm{id}, \tau_{2,2}, \dots, \tau^5_{2,2}\} \end{aligned}$$

There are four degenerate 1-simplices by evaluating $s_0(\Lambda^1_2[0])$: $\delta_0\sigma_0$ and $\delta_1\sigma_0$, and $s_0(\delta_0\tau_{2,0}) = \delta_0\tau_{2,0}\sigma_0 = \delta_0\sigma_0\tau^2_{2,1}$ and analogously $\delta_1\sigma_0\tau^2_{2,1}$. The remaining eight non-degenerate 1-simplices appear as arrows in the diagram below; their position is determined by computing their endpoints via d_0 and d_1.

For the non-degenerate 2-simplices, we have to compute $s_i(\Lambda^1_2[1])$ for $i = 0, 1$ and remove these simplices from $\Lambda^1_2[2]$. Noting that $s_0(\cdots\tau^j_{2,1}) = \cdots\tau^{j-1}_{2,1}\sigma_1\tau^2_{2,2}$ and $s_1(\cdots\tau^j_{2,1}) = \cdots\tau^{j-1}_{2,1}\sigma_0\tau_{2,2}$, plus the relation $\sigma_0\sigma_1 = \sigma_0\sigma_0$ establishes that all $\delta_i\sigma_0\sigma_0\tau^k_{2,2}$, $i = 0, 1$, $k = 0, 1, 2, 3, 4, 5$ are degenerate. Applying s_0 to $\mathrm{id}_{[1]}\,\tau^j_{2,1}$ shows that the simplices $\sigma_0, \sigma_1\tau^2_{2,2}, \sigma_0\tau^3_{2,2}$ and $\sigma_1\tau^5_{2,2}$ are degenerate; likewise for s_1, the simplices $\sigma_1, \sigma_0\tau_{2,2}, \sigma_1\tau^3_{2,2}$ and $\sigma_0\tau^4_{2,2}$ are degenerate. This leaves four non-degenerate 2-simplices: $\sigma_1\tau_{2,2}, \sigma_0\tau^2_{2,2}, \sigma_1\tau^4_{2,2}$ and $\sigma_0\tau^5_{2,2}$. ⌟

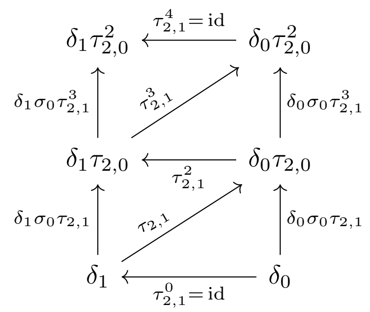


Figure 5: An explicit depiction of the standard 2-cyclic set Λ^1_2. Note that top and bottom vertices and 1-simplices are identified. The 1-simplices are drawn as arrows x, pointing from $d_0(x)$ to $d_1(x)$. The realizations of this cyclic set is $\mathbb{R}/2\mathbb{Z} \times |\Delta^1|$.

Proposition 6.10. *The following three properties establish important key facts about the standard r-cyclic sets.*

1. *The r-cyclic set $\Lambda^0_r[-]$ has r vertices, and r nondegenerate 1-simplices. All higher-dimensional simplices are degenerate. Its realization is $|\Lambda^0_r| \cong \mathbb{R}/r\mathbb{Z}$.*

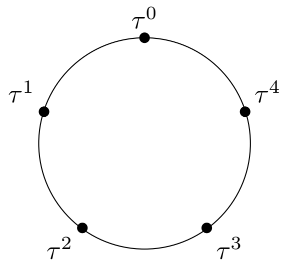
[5] An equivalent depiction of $\Lambda^1_2[-]$ also appears in [DGM12, p. 236].

2. *The standard n-dimensional r-cyclic set* $\Lambda^n_r[-]$ *has realization* $|\Lambda^n_r| \cong \mathbb{R}/r\mathbb{Z} \times |\Delta^n|$.

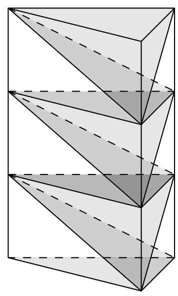
3. *There exists a* simplicial C_r*-action on* $\Lambda^n_r[-]$*, and the induced action of the discrete subgroup* $\mathbb{Z}/r\mathbb{Z} \subset \mathbb{R}/r\mathbb{Z}$ *on the realization is simplicial.*

Proof. We have $\Lambda^0_r[0] = \Delta^0[0] \times C_r$, and thus r vertices $\tau^0_{r,0}, \ldots, \tau^{r-1}_{r,0}$. Applying s_0 to these vertices, we obtain $\tau^0_{r,1}, \tau^2_{r,1}, \ldots, \tau^{2(r-1)}_{r,1}$, thus we have r non-degenerate 1-simplices $\tau^i_{r,1}$ for odd i, $1 \le i \le 2(r-1)+1$. A 1-simplex $g = \tau^{2k+1}_{r,1}$ is attached to $d_0(g) = g\delta_0 = \tau^k_{r,0}$ and $d_1(g) = g\delta_1 = \tau^{k+1}_{r,0}$. We have $d_1(\tau^{2(r-1)+1}_{r,1}) = \tau^r_{r,0} = \tau^0_{r,0}$, thus in the realization we have r copies of $|\Delta^1| \cong [0;1] \subset \mathbb{R}$ glued end-to-end along the r vertices to form a circle; see figure 6 (a) for the case $|\Lambda^0_5|$. A point $[\tau^k_{r,1}, (t, 1-t)]$ corresponds to $k + t \in [k; k+1] \subset \mathbb{R}/r\mathbb{Z}$, $k = 0, 1, \ldots, r-1$, $0 \le t \le 1$.

The proof of lemma 5.7 can be used almost word for word to prove $|\Lambda^n_r| \cong \mathbb{R}/r\mathbb{Z} \times |\Delta^n|$. Note that the cocyclic structure maps $T_{r,n}$ rotate the corners of the n-simplices, so we have $T^0_{r,n} = T^{n+1}_{r,n} = \cdots = T^{r(n+1)}_{r,n} = \mathrm{id}$. See figure 6 (b) for an illustration of the realization of the standard 2-dimensional 3-cyclic set Λ^2_3.



(a) The realization of $|\Lambda^0_5|$. The five $[0;1]$ segments are $\tau^1_{5,1}, \tau^3_{5,1}, \ldots, \tau^9_{5,1}$. They are glued in to form $\mathbb{R}/5\mathbb{Z}$, which can be identified SO(2) in the usual fashion.



(b) As before, the top and bottom dashed triangles are identified. Note in particular how the $\mathbb{Z}/3\mathbb{Z}$-action is simplicial in the realization, i. e. sending simplices to simplices. This is not the case in $|\Lambda^2|$, cf. figure 2.

Figure 6: An illustration of $|\Lambda^0_5|$ and $|\Lambda^2_3|$.

We have, by employing the cocyclic relations (2),

$$\begin{aligned}
\tau^{m+1}_{r,m}\delta_i &= \tau^m_{r,m}\delta_{i-1}\tau_{r,m-1} = \cdots = \tau^{m-i+1}_{r,m}\delta_0\tau^i_{r,m-1} \\
&= \tau^{m-i}_{r,m}\delta_m\tau^i_{r,m-1} = \delta_i\tau^m_{r,m-1}, \text{ and similarly} \\
\tau^{m+1}_{r,m}\sigma_i &= \sigma_i\tau^{m+2}_{r,m+1}.
\end{aligned}$$

Since $C_r \cong \langle t^{m+1}_{r,m} \rangle$ for all m (recall that $t^{r(m+1)}_{r,m} = \mathrm{id}$), the canonical C_r action is simplicial. The induces the canonical action of $\mathbb{Z}/r\mathbb{Z}$ on $\mathbb{R}/r\mathbb{Z} \times |\Delta^n|$, which as a result is simplicial as well. □

Corollary 6.11. *A point* $(\theta; p_0, \ldots, p_n) \in \mathbb{R}/r\mathbb{Z} \times |\Delta^n|$ *has a "quotient representation"* $[\sigma_i \circ g(\theta); q_0, \ldots, q_{n+1}]$ *with* $g(\theta) = \tau^{k(n+2)+n+2-i}_{r,n+1}$.

Proof. We choose nonnegative p', p'' with $p' + p'' = p_i$ and $k \in [r-1]$ such that $\theta = -k - p_0 - \cdots - p_{i-1} - p'$ in $\mathbb{R}/r\mathbb{Z}$, and define $(q_0, \ldots, q_{n+1}) := (p'', p_{i+1}, \ldots, p_n, p_0, \ldots, p_{i-1}, p')$. Then we calculate directly using the homeomorphism $[xg, p] \mapsto ([g, p], [x, g(p)])$ from lemma 5.7:

$$\begin{aligned}
&[\sigma_i \circ \tau^{k(n+2)+n+2-i}_{r,n+1}; q_0, \ldots, q_{n+1}] = \\
&\quad = ([\tau^{k(n+2)+n+2-i}_{r,n+1}, q_0, \ldots, q_{n+1}], [\sigma_i, p_0, \ldots, p_{i-1}, p', p'', p_{i+1}, \ldots, p_n])
\end{aligned}$$

$$= (-k - p_0 - \cdots - p_{i-1} - p'; p_0, \ldots, p_n) = (\theta; p).$$ □

Proposition 6.12. *For $n \geq 0$, there is a simplicial map $\mathrm{sd}_a : \Lambda^n_{as} \to \mathrm{sd}_a \Lambda_s^{a(n+1)-1}$.*

Proof. We map $x : [m] \to [n] \in \Delta^n[m]$ and $t \in C^{\mathsf{op}}_{as(m+1)}$ as follows:

$$\mathrm{sd}_a : \Lambda^n_{as} = \Delta^n[-] \times C^{\mathsf{op}}_{as(-+1)} \to \mathrm{sd}_a \Lambda_s^{a(n+1)-1} = \Delta^{a(n+1)-1}[\mathrm{sd}_a(-)] \times C^{\mathsf{op}}_{as(-+1)}$$
$$(x, t) \mapsto (x \sqcup \cdots \sqcup x, t),$$

where $x \sqcup \cdots \sqcup x$ denotes the "diagonal inclusion by concatenation", i. e. it is a map $[a(m+1)-1] \to [a(n+1)-1]$, cf. definition 6.3. The map is by construction simplicial, because for $f : [m'] \to [m]$ we have:

$$\begin{array}{ccc} x : [m] \to [n] & \xmapsto{\quad\mathrm{sd}_a\quad} & x \sqcup \cdots \sqcup x \\ \downarrow f & & \downarrow f \sqcup \cdots \sqcup f \\ f(x) = xf : [m'] \to [n] & \xmapsto{\mathrm{sd}_a} xf \sqcup \cdots \sqcup xf = & f \sqcup \cdots \sqcup f(x \sqcup \cdots \sqcup x) \end{array}$$ □

Proposition 6.13. *The map $|\,\mathrm{sd}_a -|$ can be described by the following diagram:*

$$\begin{array}{ccccc} |\Lambda^n_{as}| & \xrightarrow{|\,\mathrm{sd}_a -|} & |\,\mathrm{sd}_a \Lambda_s^{a(n+1)-1}| & \xrightarrow[\cong]{D_a} & |\Lambda_s^{a(n+1)-1}| \\ \| & & & & \| \\ \mathbb{R}/as\mathbb{Z} \times |\Delta^n| & & \xrightarrow{\frac{1}{a} \times d_a} & & \mathbb{R}/s\mathbb{Z} \times |\Delta^{a(n+1)-1}| \end{array}$$

Proof. The diagram is indeed commutative. To see this we need to use the quotient representation from corollary 6.11 for a point when mapping the top way. Given a point $(\theta, p) = [x \circ g(\theta), q_0, \ldots, q_{n+1}]$, with $x = \sigma_i$ and $g(\theta) = \tau_{as,n+1}^{(ka+k')(n+2)+n+2-i}$, we map:

$$\begin{array}{ccccc} [xg(\theta); q] & \longmapsto & [(x \sqcup \cdots \sqcup x)g(\theta), q] & \longmapsto & [(x \sqcup \cdots \sqcup x)g(\theta), (\frac{1}{a}q, \ldots, \frac{1}{a}q)] \\ \| & & & & \| \\ & & & & ([g(\theta); (\frac{1}{a}q, \ldots, \frac{1}{a}q)], \\ & & & & [x \sqcup \cdots \sqcup x; \tau_{s,n+1}^{n+2-i}(\frac{1}{a}q, \ldots, \frac{1}{a}q)]) \\ \| & & & & \| \\ (\theta; p) & & \xmapsto{\frac{1}{a} \times d_a} & & (\frac{\theta}{a}; \frac{1}{a}p, \ldots, \frac{1}{a}p) \end{array}$$

Note in particular that we have $\theta = -(ka + k') - p_0 - \cdots - p_{i-1} - p'$ and thus

$$\begin{aligned} [g(\theta), (\frac{1}{a}q, \ldots, \frac{1}{a}q)] &= [\tau_{as,n+1}^{(ka+k')(n+2)+n+2-i}, (\frac{1}{a}q, \ldots, \frac{1}{a}q)] \\ &= -k - k'\frac{1}{a}(q_0 + \cdots + q_n) - \frac{1}{a}(p_0 + \cdots + p_{i-1} + p') \\ &= \frac{-ka - k' - p_0 - \cdots - p_{i-1} - p'}{a} = \frac{\theta}{a} \end{aligned}$$

for the right-hand equality. □

Theorem 6.14. *Let $X_\bullet$ be an s-cyclic space. Then $D : |\,\mathrm{sd}_a X_\bullet| \to |X_\bullet|$ is an* SO(2)*-equivariant homeomorphism.*

Proof. The previous proposition lets us identify what D does on the components that are identified with $\mathrm{SO}(2)$:

$$\begin{array}{ccc}
|\operatorname{sd}_a X_\bullet| & \xrightarrow[\cong]{D} & |X_\bullet| \\
\| & & \| \\
\coprod X_{a(n+1)-1} \times |\Lambda^n_{as}| \big/ \approx & \xrightarrow{\mathrm{id} \times D|\operatorname{sd}_a -|} & \coprod X_n \times |\Lambda^n_s| \big/ \approx \\
\| & & \| \\
\coprod X_{a(n+1)-1} \times \mathbb{R}/as\mathbb{Z} \times |\Delta^n| \big/ \approx & \xrightarrow{\mathrm{id} \times \frac{1}{a} \times d_a} & \coprod X_n \times \mathbb{R}/s\mathbb{Z} \times |\Delta^n| \big/ \approx
\end{array}$$

The identification of $\mathbb{R}/as\mathbb{Z}$ and $\mathbb{R}/s\mathbb{Z}$ with $\mathrm{SO}(2)$ differ by division with a, and that is exactly what D does on this component. Thus we obtain a diagram

$$\begin{array}{ccccc}
\mathbb{R}/as\mathbb{Z} \times |\operatorname{sd}_a X_\bullet| & \xrightarrow{\mathrm{id} \times D} & \mathbb{R}/as\mathbb{Z} \times |X_\bullet| & \xrightarrow{\frac{1}{a} \times \mathrm{id}} & \mathbb{R}/s\mathbb{Z} \times |X_\bullet| \\
\downarrow{\scriptstyle \text{action}} & & & & \downarrow{\scriptstyle \text{action}} \\
|\operatorname{sd}_a X_\bullet| & & \xrightarrow{D} & & |X_\bullet|
\end{array}$$

showing the equivariance of the homeomorphism. □

The third assertion in proposition 6.10 together with this equivariance theorem imply the following important result: Given the realization of a cyclic space $X_\bullet$, we can analyze its fixed points under a finite subgroup C of $\mathrm{SO}(2)$ by looking at the corresponding simplicial action in a suitable subdivision. Formally:

Corollary 6.15. *Let $C_a \subset \mathrm{SO}(2)$ be a finite cyclic subgroup of order a. Then $|X_\bullet|^{C_a} \cong |(\operatorname{sd}_a X_\bullet)^{C_a}|$.*

Proof. By theorem 6.14, D_a is equivariant with respect to the $\mathrm{SO}(2)$-, and thus C_a-action, and the fixed point space can be described as an equalizer (cf. section 3). Thus we have: $|(\operatorname{sd}_a X_\bullet)^{C_a}| \cong |\operatorname{sd}_a X_\bullet|^{C_a} \overset{D}{\cong} |X_\bullet|^{C_a}$. □

Proposition 6.16. *The subdivision functor can be iterated with $\operatorname{sd}_a \operatorname{sd}_b = \operatorname{sd}_{ab}$, and the diagram*

$$\begin{array}{ccc}
|\operatorname{sd}_{ab} X_\bullet| & \xrightarrow{D_{ab}} & |X_\bullet| \\
& {\scriptstyle D_a} \searrow \quad \nearrow {\scriptstyle D_b} & \\
& |\operatorname{sd}_b X_\bullet| &
\end{array}$$

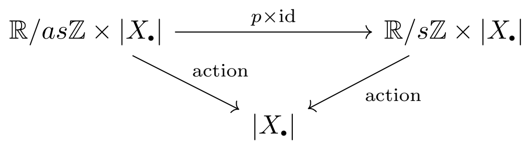
commutes, i. e. $D_b D_a = D_{ab}$.

Proof. Commutativity of the diagram follows from proposition 6.13 and observing that $d_b d_a = d_{ab}$. □

Definition 6.17. For $a \geq 1$, the functor $P_a : \Lambda_{as} \to \Lambda_s$ is the identity on objects and morphisms inherited from Δ, and projects $\tau_n^{i(n+1)+r} \mapsto \tau_n^r$. ⌟

Remark 6.18. Since P_a is the identity on objects, unlike with the sd_a functor, the n-simplices of $X_\bullet$ and $P_a X_\bullet$ are equal. What changes is that for an s-cyclic space $X_\bullet$, $P_a X_\bullet$ carries a C_{as}-action. If the context is understood one can drop the P_a and say $X_\bullet$ carries both a C_s- and C_{as}-action. How these compare to one another in the realization will be stated in the following proposition. ⌟

Proposition 6.19. *Let $X_\bullet$ be a Λ_s-space, and $p : \mathbb{R}/as\mathbb{Z} \to \mathbb{R}/s\mathbb{Z}$ induced from the identity map on $\mathbb{R}$. Then $|X_\bullet|$ carries an induced $\mathbb{R}/s\mathbb{Z}$ and $\mathbb{R}/as\mathbb{Z}$ action, and the following diagram commutes:*



Proof. In the realization, the map $P_a : \Lambda^n_{as} \to \Lambda^n_s$ induces $|P_a| = p \times \mathrm{id} : \mathbb{R}/as\mathbb{Z} \times |\Delta^n| \to \mathbb{R}/s\mathbb{Z} \times |\Delta^n|$. For a general cyclic space one argues like the proof of theorem 6.14. □

7 The Cyclic Nerve of a Group

An interesting object of study are classifying spaces of groups. Underlying this concept is the more general concept of the (simplicial) nerve of a category and its realization. The related concept of the cyclic nerve, which for topological groups is a cyclic space, is our main focus: Its realization inherits a canonical circle action, and can be used to model (up to homotopy) the free loop space of a group's classifying space.

Definition 7.1. Let C be a (locally small) category. Define $i : \Delta \to \mathsf{Cat}$ by sending $[n] \mapsto \{0 \to 1 \to \cdots \to n\}$, which shall denote the category corresponding to the totally ordered set $[n]$; morphisms are functors that respect the ordering. The *(simplicial) nerve* of C is defined as the functor category $N_\bullet(\mathsf{C}) = \mathrm{Hom}_{\mathsf{Cat}}(i-, \mathsf{C})$. Concretely, for $n \in \mathbb{N}$, $N_n(\mathsf{C}) = \{f_n \circ f_{n-1} \circ \cdots \circ f_1 | f_i \in \mathrm{Mor}(\mathsf{C}) \text{ composable}\}$.

Similarly, the inclusion $i_{cy} : \Lambda \to \mathsf{Cat}$ sending $[n] \mapsto \{0 \to 1 \to \cdots \to n \to 0\}$ gives rise to the *cyclic nerve* $N^{\mathrm{cy}}_\bullet(\mathsf{C}) = \mathrm{Hom}_{\mathsf{Cat}}(i_{cy}-, \mathsf{C})$. Concretely, the cyclic nerve has the additional restriction that the $n+1$ composable morphisms in $N^{\mathrm{cy}}_n(\mathsf{C})$ must also allow a cyclic permutation of the order, i. e. $f_1 \circ f_{n+1}$ must be well-defined. ⌟

Proposition 7.2. *A nerve $N_\bullet(\mathsf{C}) : \Delta^{\mathsf{op}} \to \mathsf{Set}$ is a simplicial set.*

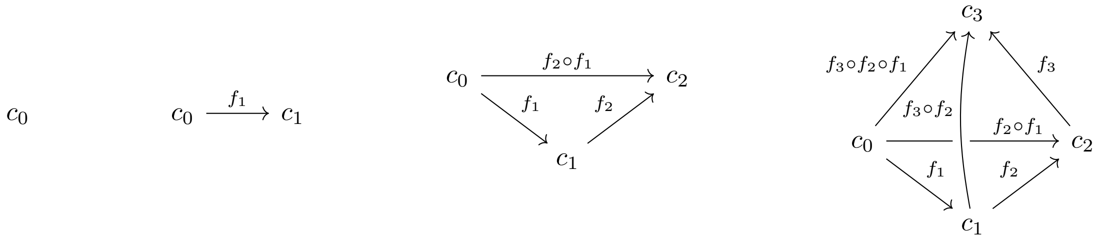
Proof. We describe the face and degeneracy maps explicitly:

$$
\begin{aligned}
d_i(f_n \circ \cdots \circ f_1) &= \begin{cases} f_n \circ \cdots \circ f_2 & i = 0 \\ f_n \circ \cdots \circ (f_{i+1} \circ f_i) \circ \cdots \circ f_1 & 0 < i < n \\ f_{n-1} \circ \cdots \circ f_1 & i = n \end{cases} \\
s_i(f_n \circ \cdots \circ f_1) &= f_n \circ \cdots \circ f_{i+1} \circ \mathrm{id} \circ f_i \circ \cdots \circ f_1 \quad 0 \le i \le n
\end{aligned}
$$

The maps $d_i : N_n(\mathsf{C}) \to N_{n-1}(\mathsf{C})$ and $s_i : N_n(\mathsf{C}) \to N_{n+1}(\mathsf{C})$ are well-defined by the composition laws for categories. It is easy to check that the relations (1^{op}) are safisfied, e. g. for $0 \le i < j \le n$ one has:

$$
\begin{aligned}
d_i s_j(f_n \cdots f_1) &= d_i(f_n \cdots f_{j+1} \,\mathrm{id}\, f_j \cdots f_1) = f_n \cdots f_{j+1} \,\mathrm{id}\, f_j \cdots (f_{i+1} f_i) \cdots f_1 \\
&= s_{j-1}(f_n \cdots f_{j+1} f_j \cdots (f_{i+1} f_i) \cdots f_1) = s_{j-1} d_i(f_n \cdots f_1).
\end{aligned}
$$
□

Remark 7.3. One can actually see simplex-like shapes in the diagrams that constitute $N_\bullet(\mathsf{C})$ in low dimensions: The vertices $N_0(\mathsf{C})$ are empty composition of arrows, hence amount to choosing objects $c_i \in \mathrm{Ob}(\mathsf{C})$; the 1-cells are morphisms in C; the two-cells are commutative tringles; and the three-cells are pyramid-shaped commutative diagrams:



One can easily see how the face and degeneracy operations work in these diagrams just like they do on the standard n-simplex. ⌟

Nerves and realization have a deeper link than one would expect at first. Namely, they are both adjoints to functors to/from simplicial sets, as will be made precise in the corollaries of the following

Proposition 7.4. *Let* C *be a (locally small, cocomplete) category. Then a cosimplicial object* $\Sigma : \Delta \to \mathsf{C}$ *determines a unique adjoint functor pair* (L, R) *with* $\mathsf{C}(LX, Y) \cong \mathsf{S}(X, RY)$.

Proof. Let Σ be an cosimplicial object in C. Then, for a simplicial set $X_\bullet$, we define $L : \mathsf{S} \to \mathsf{C}$ as the coend

$$LX := \int^m X_m \times \Sigma^m,$$

and $f_\bullet : X_\bullet \to X'_\bullet$ translates to $Lf_\bullet : LX_\bullet \to LX'_\bullet$. By the universal property of the coend and because Σ is a functor, L is a functor.

Define $RY := \mathsf{C}(\Sigma-, Y)$; more concretely, given $Y \in \mathrm{Ob}(\mathsf{C})$, set $RY_n = \mathsf{C}(\Sigma^n, Y)$. Maps $f : Y \to Y'$ translate to postcomposition. This makes RY a functor $\Delta^{\mathsf{op}} \to \mathsf{Set}$, a simplicial set; the face and degeneracy maps are determined by precomposing with their image under Σ.

By the Yoneda lemma $\mathsf{S}(\Delta^n, RY) \cong RY_n = \mathsf{C}(\Sigma^n, Y)$, and analogously to the proof of theorem 2.8 it follows $\mathsf{C}(LX, Y) \cong \mathsf{S}(X, RY)$. □

Corollary 7.5. *The cosimplicial object* $|\Delta^\bullet| : \Delta \to \mathsf{Top}$ *determines the functor pair "realization and singular set", in symbols:* $|-| \dashv \mathrm{Sing}$. *(See also example 1.5, proposition 2.6 and theorem 2.8.)*

Corollary 7.6. *The cosimplicial object* $i : \Delta \to \mathsf{Cat}$ *from definition 7.1 determines as right-adjoint the nerve functor; that is,* $\mathsf{Cat}(iX_\bullet, \mathsf{C}) \cong \mathsf{S}(X_\bullet, N_\bullet\mathsf{C})$.

Definition 7.7. Let C be a (locally small, cocomplete) category. Then the *realization of* C is defined via the composition $\mathsf{Cat} \xrightarrow{N} \mathsf{S} \xrightarrow{|-|} \mathsf{Top}$. Usually, $|N(\mathsf{C})|$ is called the *classifying space of* C and denoted $B\mathsf{C}$. ⌟

We now turn to the cyclic nerve and its relation to the simplicial nerve. Certainly every cyclic nerve is a simplicial nerve by truncation:

$$\pi : N_n^{\mathrm{cy}}\mathsf{C} \to N_n\mathsf{C}, \quad f_n \cdots f_0 \mapsto f_n \cdots f_1.$$

If C is a groupoid, i. e. a category where all morphisms are automorphisms, then the simplicial nerve admits an embedding into the cyclic nerve:

Proposition 7.8. *Let* C *be a groupoid. Then there exists an embedding* $i : N_\bullet\mathsf{C} \to N_\bullet^{\mathrm{cy}}\mathsf{C}$. *Furthermore,* $\pi \circ i = \mathrm{id}$.

Proof. Let $f_n \circ \cdots \circ f_1$ be an n-simplex in $N_n\mathsf{C}$. Since all morphisms are invertible, we can set $f_0 := f_1^{-1} \circ \cdots \circ f_n^{-1} = (f_n \circ \cdots \circ f_1)^{-1}$, and then $f_n \circ \cdots \circ f_1 \circ f_0$ is an element of $N_n^{\mathrm{cy}}\mathsf{C}$, since by construction $\mathrm{cod}(f_0) = \mathrm{dom}(f_n)$. □

We recall the standard categorical construction of a group (or, more generally, a monoid) as the hom-set of a single-object category:

Definition 7.9. Given a monoid M, then the category with a single element $*$ and morphism set M is denoted by M. Multiplication of elements in M correspond to composition in M; in case M is a group, all morphisms are automorphisms. ⌟

Remark 7.10. Given a group G, the nerve of the groupoid G is $N_\bullet\mathsf{G} \cong G^{\times\bullet}$, where G^0 is the sole object of G. Note that the restriction of "composability of arrows" is no restriction in the categorical construction of a group (or monoid), since all morphisms are composable. Thus we refer to elements of $N_n\mathsf{G}$ by an n-tuple $(g_1, \ldots, g_n)$; in particular note that the order is reversed, as we regard this as tuples of elements that act by multiplication on the right.

Similarly, the cyclic nerve of G is $N_n^{\text{cy}}\mathsf{G} \cong G^{n+1}$. In light of this notation, the inclusion map from proposition 7.8 reads as follows:

$$i : N_n\mathsf{G} \to N_n^{\text{cy}}\mathsf{G}, \quad (g_1, \ldots, g_n) \mapsto ((g_1 \cdots g_n)^{-1}, g_1, \ldots, g_n).$$

The cyclic nerve $N_\bullet^{\text{cy}}\mathsf{G}$ inherits a canonical cyclic structure, given by

$$t_n : (g_0, \ldots, g_n) \mapsto (g_n, g_0, \ldots, g_{n-1}).$$

⌟

The nerve construction via Hom-sets again has the limitation that it cannot take into account additional structure like a topology (compare remark 2.9). We therefore define the nerve functor for topological groups in such a way that for discrete groups it agrees with the nerve of the one-element groupoid construction.

Definition 7.11. Let G be a topological group. Define the nerve functor $N_\bullet : \mathsf{TopGrp} \to \mathsf{SimpSpace}$ via $N_n(G) = G^n$ with the product topology and the obvious structure maps (cf. prop. 7.2). The cyclic nerve $N_\bullet^{\text{cy}} : \mathsf{TopGrp} \to \mathsf{CycSpace}$ is given via $N_n^{\text{cy}}(G) = G^{n+1}$, and structure maps as in the next proposition. ⌟

Proposition 7.12. *Given a topological monoid G, $N_\bullet^{\text{cy}}G$ is a cyclic space.*

Proof. This result holds by construction, but for clarity we write down explicitly the structure maps: We have $N_n^{\text{cy}}G = G^{n+1}$ with the product topology induced by the topology carried by G, and

$$d_i(g_0, \ldots, g_n) = \begin{cases} (g_0g_1, g_2, \ldots, g_n) & i = 0 \\ (g_0, \ldots, g_ig_{i+1}, \ldots, g_n) & 0 < i < n \\ (g_ng_0, g_1, \ldots, g_{n-1}) & i = n \end{cases}$$
$$s_i(g_0, \ldots, g_n) = (g_0, \ldots, g_i, e_G, g_{i+1}, \ldots, g_n)$$
$$t_n(g_0, \ldots, g_n) = (g_n, g_0, \ldots, g_{n-1})$$

Since multiplication is continuous in G, all structure maps are continuous. □

It might be interesting to see that not only the cyclic nerve admits a cyclic structure:

Proposition 7.13. *Let G be a topological group. Then $N_\bullet(G)$ admits a cyclic space structure.*

Proof. Since $N_nG = G^n$, a pure "rotation of coordinates" as in the cyclic nerve case is not possible. The face and degeneracy maps are as in proposition 7.2, and t_n is given by

$$t_n : G^n \to G^n, \quad (g_1, \ldots, g_n) \mapsto (z(g_1 \cdots g_n)^{-1}, g_1, \ldots, g_{n-1}),$$

where z is any element in the center of G (e. g. the unit element 1). The map is continuous, and the cyclic relations (2^{op}) hold, in particular:

$$\begin{aligned}
t_n^{n+1}(g_1, \ldots, g_n) &= t_n^n(zg_n^{-1}\cdots g_1^{-1}, g_1, \ldots, g_{n-1}) \\
&= t_n^{n-1}(zg_{n-1}^{-1}\cdots g_1^{-1}g_1\cdots g_n z^{-1}, zg_n^{-1}\cdots g_1^{-1}, g_1, \ldots, g_{n-2}) \\
&= t_n^{n-1}(g_n, zg_n^{-1}\cdots g_1^{-1}, g_1, \ldots, g_{n-2}) \\
&= t_n^{n-2}(zg_{n-2}^{-1}\cdots g_1^{-1}g_1\cdots g_n z^{-1}g_n^{-1}, g_n, zg_n^{-1}\cdots g_1^{-1}, g_1, \ldots, g_{n-3}) \\
&= t_n^{n-2}(g_{n-1}, g_n, zg_n^{-1}\cdots g_1^{-1}, g_1, \ldots, g_{n-3}) \\
&\ldots \\
&= t_n(g_2, \ldots, g_n, zg_n^{-1}\cdots g_1^{-1}) \\
&= (zg_1\cdots g_n z^{-1}g_n^{-1}\cdots g_2^{-1}, g_2, \ldots, g_n) \\
&= (g_1, g_2, \ldots, g_n).
\end{aligned}$$

It is essential that z commute with other group elements for $t_n^{n+1} = \mathrm{id}$ to hold; the other cyclic relations are straightforward to check. Thus we endow $N_\bullet(G)$ with a cyclic space structure; Loday calls this construction the *twisted nerve of* G, see [Lod98, 7.3.3]. □

For the rest of the section, let G be a topological group. We apply results from the previous sections to follow [BHM93, section 2].

Definition 7.14. The diagonal map on the cyclic nerve of G is

$$\Delta_{a,\bullet} : N_\bullet^{\mathrm{cy}}(G) \to \mathrm{sd}_a\, N_\bullet^{\mathrm{cy}}(G), \quad (g_0, \ldots, g_n) \mapsto (g_0, \ldots, g_n, \ldots, g_0, \ldots, g_n).$$

We denote by Δ_a the realization $|\Delta_{a,\bullet}|$. ⌟

Proposition 7.15. *When restricting the codomain, the diagonal map is a simplicial isomorphism* $\Delta_{a,\bullet} : N_\bullet^{\mathrm{cy}}(G) \to (\mathrm{sd}_a\, N_\bullet^{\mathrm{cy}}(G))^{C_a}$

Proof. The C_a-action is generated by $t_{a(n+1)-1}^{n+1}$, and the inverse map is truncation to the non-repeating sequence. □

Proposition 7.16. *The map* $\Delta_{a,\bullet} : P_a N_\bullet^{\mathrm{cy}}(G) \to \mathrm{sd}_a\, N_\bullet^{\mathrm{cy}}(G)$ *is a morphism of* Λ_a^{op}*-spaces.*

Proof. Recall the definition of P_a from 6.17: $P_a N_\bullet^{\mathrm{cy}}(G)$ is the space $N_\bullet^{\mathrm{cy}}(G)$ with a $C_{a(n+1)+1}$-action on the n-simplices, generated by t_n. We know $\Delta_{a,\bullet}$ is simplicial, so it remains to show it is a-cyclic: Let $\bar{g} = (g_0, \ldots, g_n) \in N_\bullet^{\mathrm{cy}}(G)$ and $\bar{g'} = (g_n, g_0, \ldots, g_{n-1})$. Then:

$$t_n \Delta_{a,n}(\bar{g}) = t_n(\bar{g}, \ldots, \bar{g}) = (\bar{g'}, \ldots, \bar{g'}) = \Delta_{a,n}(\bar{g'}) = \Delta_{a,n} t_n(\bar{g}).$$ □

Proposition 7.17. *There following diagram is commutative up to homotopy:*

$$\begin{array}{ccccc}
|N_\bullet(G)| & & \xrightarrow{\quad |i| \quad} & & |N_\bullet^{\mathrm{cy}}(G)| \\
\big\downarrow {\scriptstyle |i|} & & & & \big\uparrow {\scriptstyle D_a} \\
|N_\bullet^{\mathrm{cy}}(G)| & \xrightarrow{\Delta_a} & |\,\mathrm{sd}_a\, N_\bullet^{\mathrm{cy}}(G)|^{C_a} & \hookrightarrow & |\,\mathrm{sd}_a\, N_\bullet^{\mathrm{cy}}(G)|
\end{array}$$

Proof. We provide an explicit homotopy $H_{a,t} : D_a \Delta_a |i| \simeq |i|$. Let $(g_1, \ldots, g_n) \in N_\bullet(G)$. Then the image $[(g_0, \ldots, g_n), p] \in |N_\bullet^{\mathrm{cy}}(G)|$ has $g_0 = (g_1 \cdots g_n)^{-1}$. Applying $\Delta_a D_a$, we obtain

$$[(g_0, \ldots, g_n, \ldots, g_0, \ldots, g_n), (\frac{1}{p}, \ldots, \frac{1}{p})].$$

Let $h_{a,t} : |\Delta^n| \times [0,1] \to |\Delta^{a(n+1)-1}|$ be given by

$$(p_0, \dots, p_n; t) = (p; t) \mapsto \left(\frac{t}{a} p, \dots, \frac{t}{a} p, \left(\frac{t}{a} p + (1-t) p \right) \right).$$

Then it is clear that $h_{a,1}$ is the map d_a from lemma 6.6. In the case $t = 0$ we have $h_{a,0} = D_0^{(n+1)(a-1)}$, the repeated application of the coface map in $|\Delta^\bullet|$. Certainly $h_{a,t}$ is continuous, and $\mathrm{id} \times h_{a,t}$ induces a homotopy

$$H_{a,t} : |\operatorname{sd}_a N^{\mathrm{cy}}_\bullet(G)| \times [0,1] \to |N^{\mathrm{cy}}_\bullet(G)|$$

with $H_{a,1} = D_a$. The map $H_{a,0}$ is induced by $\mathrm{id} \times D_0^{(n+1)(a-1)}$, or equivalently by $d_0^{(n+1)(a-1)} \times \mathrm{id}$. Since $g_0 \cdots g_n = e_G$, we have

$$\begin{aligned} d_0^{(n+1)(a-1)}(g_0, \dots, g_n, \dots, g_0, \dots, g_n) &= ((g_0 \cdots g_n)^{a-1} g_0, g_1, \dots, g_n) \\ &= (g_0, g_1, \dots, g_n), \end{aligned}$$

or for short $d_0^{(n+1)(a-1)} \Delta_{a,\bullet} = \mathrm{id}$. Thus thus we conclude $H_{a,0} \Delta_a |i| = |i|$. □

Corollary 7.18. *The following diagram is homotopy-commutative:*

$$\begin{array}{ccccc} |N_\bullet(G)| & \xrightarrow{|i|} & |N^{\mathrm{cy}}_\bullet(G)| & \xrightarrow{\Delta_s} & |\operatorname{sd}_s N^{\mathrm{cy}}_\bullet(G)|^{C_s} \\ \downarrow{\scriptstyle |i|} & & & & \uparrow{\scriptstyle D_a} \\ |N^{\mathrm{cy}}_\bullet(G)| & \xrightarrow{\Delta_{as}} & |\operatorname{sd}_{as} N^{\mathrm{cy}}_\bullet(G)|^{C_{as}} & \hookrightarrow & |\operatorname{sd}_{as} N^{\mathrm{cy}}_\bullet(G)|^{C_s} \end{array}$$

Proof. The homotopy is $\Delta_s H_{as,t} : \Delta_s D_a \Delta_a |i| \simeq \Delta_s |i|$. □

Corollary 7.19. *The maps $|\pi|, |\pi| D_a \Delta_a : |N^{\mathrm{cy}}_\bullet(G)| \to |N_\bullet(G)|$ are homotopic.*

Proof. We have $\pi = \pi d_0^{(n+1)(a-1)} \Delta_{a,\bullet}$, so the desired homotopy is $|\pi| H_{a,t} \Delta_a$. □

Proposition 7.20. *Let $\bar{\Delta}_a$ be the composition*

$$\bar{\Delta}_a : |P_a N^{\mathrm{cy}}_\bullet(G)| = |N^{\mathrm{cy}}_\bullet(G)| \xrightarrow{\Delta_a} |\operatorname{sd}_a N^{\mathrm{cy}}_\bullet(G)|^{C_a} \xrightarrow{D_a} |N^{\mathrm{cy}}_\bullet(G)|^{C_a}.$$

Then $\bar{\Delta}_a$ is a homeomorphism and equivariant in the sense that $\bar{\Delta}_a(az \cdot x) = z \cdot \bar{\Delta}_a(x)$.

Proof. The map Δ_a is a homeomorphism by prop. 7.15, and composition with D_a yields a homeomorphism again. By virtue of P_a we have a $\mathbb{R}/a\mathbb{Z}$-action on both $|N^{\mathrm{cy}}_\bullet(G)|$ resp. $|\operatorname{sd}_a N^{\mathrm{cy}}_\bullet(G)|$, and by prop. 7.16, Δ_a is equivariant with respect to this action. Theorem 6.14 guarantees the equivariance of D_a in the sense that the $\mathbb{R}/a\mathbb{Z}$-action becomes a $\mathbb{R}/\mathbb{Z}$-action by "rescaling" (division by a). Thus we have for $z \in \mathbb{R}/\mathbb{Z}$ and $x \in |N^{\mathrm{cy}}_\bullet(G)|$:

$$\bar{\Delta}_a(az \cdot x) = D_a \Delta_a(az \cdot x) = D_a(az \cdot \Delta_a(x)) = z \cdot D_a \Delta_a(x) = z \cdot \bar{\Delta}_a(x).$$ □

Definition 7.21. Let G be a topological group.

i) The *classifying space of* G is $BG := |N_\bullet(G)|$.

ii) The *total space of the universal principal bundle of* G is the space EG obtained as the realization of the simplicial space $E_\bullet G = N_\bullet(G//G)$, the nerve of the G-action groupoid on G. We have $E_n G := G \times G^n$ with the usual simplicial structure maps.

iii) Given a G-space X, the *Borel construction* is the realization of the simplicial space $X \times_G E_\bullet G \cong N_\bullet(X//G)$.

iv) The *(unbased) free loop space of BG* is $LBG := \mathrm{Map}(S^1, BG)$ with the compact-open topology.

⌟

Proposition 7.22. *We collect a few well-known properties of these constructions:*

1. *EG is a contractible space with a free G-action via $g(g_0, \ldots, g_n) = (gg_0, \ldots, gg_n)$, and $BG = EG/G$.*
2. *$|X \times_G E_\bullet G| \cong X \times_G EG = (X \times EG)/G$*

Proof. See example 1B.7 and p. 322f. in [Hat02]. □

Proposition 7.23. *Let $q : |N^{\mathrm{cy}}_\bullet(G)| \to LBG$ be the adjoint of the composite map*

$$\mathrm{SO}(2) \times |N^{\mathrm{cy}}_\bullet(G)| \xrightarrow{\text{action}} |N^{\mathrm{cy}}_\bullet(G)| \xrightarrow{|\pi|} |N_\bullet G| = BG.$$

Then q is an SO(2)*-equivariant map and a homotopy equivalence.*

Proof. The map q into the free loop space of BG is defined via

$$q : |N^{\mathrm{cy}}_\bullet(G)| \to LBG, \quad p \mapsto (z \mapsto |\pi|(zp)),$$

where the action of z on p is understood to be the canonical SO(2)-action of a cyclic space (see theorem 5.9). The SO(2)-action on LBG is "rotation of the loops":

$$z' \cdot q(p) = z' \cdot (z \mapsto |\pi|(zp')) = (z \mapsto |\pi|(zz'p')) = q(z'p),$$

thus q is an equivariant map. It is also a homotopy equivalence via the composition

$$|N^{\mathrm{cy}}_\bullet(G)| \xrightarrow[\cong]{f} G^{\mathrm{ad}} \times_G EG \xrightarrow[\sim]{h} LBG.$$

The space G^{ad} is G with G-action conjugation, i. e. $\mathrm{Hom}_{G^{\mathrm{ad}}}(x, x') = \{g \mid g^{-1}xg = x'\}$. The homeomorphism f is induced by the map

$$\begin{aligned} \tilde{f} : N^{\mathrm{cy}}_\bullet(G) \cong G^{n+1} &\to G^{\mathrm{ad}} \times_G E_\bullet G = N_\bullet(G^{\mathrm{ad}}) \\ (g_0, \ldots, g_n) &\mapsto g_1 \cdots g_n g_0 [g_1 | \cdots | g_n], \end{aligned}$$

which is a simplicial isomorphism with inverse $g[g_1|\cdots|g_n] \mapsto ((g_1 \cdots g_n)^{-1} g_0, g_1, \ldots, g_n)$. Indeed, given $\bar{g} = (g_0, \ldots, g_n)$, one calculates

$$\begin{aligned} d_0 \tilde{f}(\bar{g}) &= g_1^{-1} g_1 \cdots g_n g_0 g_1 [g_2 | \cdots | g_n] = \tilde{f} d_0(\bar{g}) \quad \text{and} \\ d_n \tilde{f}(\bar{g}) &= (g_1 \cdots g_{n-1}) g_n g_0 [g_1 | \cdots | g_{n-1}] = \tilde{f} d_n(\bar{g}), \end{aligned}$$

and the other equalities (1^{op}) are trivially satisfied.

The homotopy equivalence h is best understood by "looking backwards": a loop $\lambda : [0; 1] \to BG$ corresponds to a tuple $(g, \tilde{\lambda}(0))$, where $\tilde{\lambda}$ is a lift of λ along the fibration $EG \to BG$, and g is uniquely determined such that $\lambda(0)g = \lambda(1)$. Lemma 2.12.1 of [Ben98] gives a proof that this is indeed a well-defined homotopy equivalence; also see [Mad95, p. 202]. □

Corollary 7.24. *Under q, the map $|i|$ is the inclusion of trivial loops $BG \hookrightarrow LBG$.*

Proof. We have $\tilde{f}i(g_1, \ldots, g_n) = e[g_1|\cdots|g_n]$. Thus an element $z \in BG$ correspond to the map $\tilde{\lambda} : [0; 1] \to EG$ with $\tilde{\lambda}([0; 1]) = \{z\} \subset EG$, and the projection $\lambda \in LBG$ is the trivial loop. □

Theorem 7.25. *For a (topological) group G and each finite subgroup $C \subset \mathrm{SO}(2)$,*

$$q^C : |N_\bullet^{\mathrm{cy}}(G)|^C \to (LBG)^C$$

is a homotopy equivalence.

Proof. Every finite subgroup C of SO(2) can be identified with a finite cyclic group. Let $a \geq 1$ such that $C \cong C_a$.

Proposition 7.23 already established that q is a homotopy equivalence. We will show here that q^{C_a} is homotopic to q, which implies it is also a homotopy equivalence.

Let $R_a : LBG \to LBG$ be given by sending a loop $\lambda : z \mapsto z \cdot x$ to $R_a(\lambda) : z \mapsto az \cdot x$, i. e. running with a-fold speed around a times. Restricting the codomain, we have an obvious homeomorphism $R_a : LBG \to (LBG)^{C_a}$.

We can factor q as the composition

$$q : |N_\bullet^{\mathrm{cy}}(G)| \xrightarrow{g} L|N_\bullet^{\mathrm{cy}}(G)| \xrightarrow{L|\pi|} LBG,$$

where g is the adjoint to the SO(2)-action, and $L|\pi|$ is the "loop-ing" of the realized projection map. Then there exists a commutative diagram with all vertical maps homeomorphisms:

$$\begin{array}{ccccc} |N_\bullet^{\mathrm{cy}}(G)|^{C_a} & \xrightarrow{g^{C_a}} & (L|N_\bullet^{\mathrm{cy}}(G)|)^{C_a} & \xrightarrow{L|\pi|} & (LBG)^{C_a} \\ \bar{\Delta}_a \uparrow & & R_a \uparrow & & R_a \uparrow \\ |N_\bullet^{\mathrm{cy}}(G)| & \xrightarrow{g} & (L|N_\bullet^{\mathrm{cy}}(G)|) & \xrightarrow{L|\pi|} & LBG \end{array}$$

The left quadrilateral is commutative, because by prop. 7.20 given a loop $(\lambda(x) : z \mapsto z \cdot x) \in L|N_\bullet^{\mathrm{cy}}(G)|$, we have $\lambda(\bar{\Delta}_a(x))(z) = \lambda(x)(az) = R_a(\lambda(x))(z)$.

By corollary 7.19, the maps $|\pi|$ and $|\pi|\bar{\Delta}_a$ are homotopic, which induces homotopies via the loop functor L. Therefore the right quadrilateral is homotopy-commutative, which finishes the proof. ◻

Note. The theorem's statement is false for $C = \mathrm{SO}(2)$, because while q is an SO(2)-equivariant homotopy equivalence, it is *not* in general an equivariant weak equivalence of SO(2)-spaces, which would require $\pi_i(|N_\bullet^{\mathrm{cy}}(G)|^{\mathrm{SO}(2)}, x_0) \cong \pi_i(LBG^{\mathrm{SO}(2)}, x_0)$ for all i and base points x_0. But by proposition 6.1, on the one hand we have

$$|N_\bullet^{\mathrm{cy}}(G)|^{\mathrm{SO}(2)} = \{g \in N_0(G) \cong G \mid s_0 g = t_1 s_0 g\} = \{e_G\},$$

but on the other hand the SO(2)-equivariant loops in LBG are the trivial loops. Thus $LBG^{\mathrm{SO}(2)} \cong BG$, a space which (by design!) is in general not contractible.

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The nLab (`http://ncatlab.org`) also provided interesting examples and valuable pointers to references.